

The Yang-Mills heat equation with finite action *

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Abstract

The existence and uniqueness of solutions to the Yang-Mills heat equation is proven over \mathbb{R}^3 and over a bounded open convex set in \mathbb{R}^3 . The initial data is taken to lie in the Sobolev space of order one half, which is the critical Sobolev index for this equation over a three dimensional manifold. The existence is proven by using the Zwanziger-Donaldson-Sadun method. This consists in solving first an augmented, strictly parabolic equation and then gauge transforming the solution to a solution of the Yang-Mills heat equation itself. The gauge functions needed to carry out this procedure lie in a gauge group whose informal Lie algebra consists of functions lying in the Sobolev space of index three halves. Since the supremum norm of such functions is not controlled by this Sobolev norm, the nature of this group must be itself understood in order to carry out the reconstruction procedure. Properly defined, this group is shown to be a complete topological group in its natural metric. Solutions to the Yang-Mills

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heat equation are shown to be strong solutions modulo these gauge functions. Energy inequalities and Neumann domination inequalities are used to establish needed initial behavior properties of solutions to the augmented equation.

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1 Introduction

The Yang-Mills heat equation is a weakly parabolic, quasi-linear differential equation for a Lie algebra valued 1-form on \mathbb{R}^n . Denote by \mathfrak{k} the Lie algebra of a compact Lie group K . Let

$$A(x, t) = \sum_{j=1}^n A_j(x, t) dx^j, \quad x \in \mathbb{R}^n, \quad t \geq 0, \quad (1.1)$$

with coefficients $A_j(x, t) \in \mathfrak{k}$. The Yang-Mills heat equation has the form

$$\frac{\partial}{\partial t} A(x, t) = -d^* dA(x, t) + (\text{quadratic terms} + \text{cubic terms}) \text{ in } A. \quad (1.2)$$

The linear terms are missing a portion of the Laplacian, $-\Delta = d^*d + dd^*$, on 1-forms. For this reason the equation is only weakly parabolic. This paper is concerned with the question of existence and uniqueness of solutions to the Cauchy problem for (1.2) with fairly rough initial data: Let A_0 be a \mathfrak{k} valued 1-form on \mathbb{R}^n . We seek solutions to (1.2) such that

$$A(0) = A_0. \tag{1.3}$$

There is a standard approach to the problem of existence of solutions to a quasi-linear parabolic equation, $\partial u / \partial t = (Lu)(t) + F(u(t))$, wherein L is an elliptic linear operator and F is a possibly non-linear function of the unknown $u|_t$. One converts the differential equation to the more or less equivalent integral equation $u(t) = e^{tL}u_0 + \int_0^t e^{(t-s)L}F(u(s))ds$ and uses then a contraction principle in a space of paths $u : [0, T] \rightarrow$ 1-forms on \mathbb{R}^n . But if L is not elliptic then the basis for all the estimates that one needs in order to carry out this procedure breaks down. In the case of (1.2), one has $L = -d^*d$ on 1-forms and L is therefore not elliptic. The failure of this standard method is accompanied by the failure of the equation itself to smooth out initial data. This is quite visible in (1.2) in case the compact group K is just the circle group. In this case all the nonlinear terms are zero. The resulting equation has time independent solutions of the form $A(t, x) = d\lambda(x)$ for any function $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$. λ does not even have to be differentiable for this to be a solution because $d^2 = 0$ in any reasonable generalized sense. But if, say, $\lambda \in C_c^2(\mathbb{R}^n)$ then this is a classical solution and clearly the evolution does not smooth the initial data $A_0 := d\lambda$. Even if K is not commutative such “pure gauge” solutions exist and are similarly propagated by the equation in a time independent way. This phenomenon greatly affects the nature of the solutions that one would expect if the equation were parabolic. Ignoring this for a moment, one can compute that the critical Sobolev index for our problem in dimension three is one half in the sense of scaling. That is, the Sobolev $H_a(\mathbb{R}^3)$ norm of a 1-form is invariant under the scaling $\mathbb{R}^3 \ni x \mapsto cx$ if (and only if) $a = 1/2$, while the equation itself is invariant under the scaling $x, t \mapsto cx, c^2t$. Since we will be concerned only with spatial dimension $n = 3$, one can hope, at best, that the Cauchy problem has long time solutions when the initial data A_0 is in the Sobolev space $H_{1/2}$. This is indeed what we will prove. But the fact that some data propagates in a time independent way means that one cannot expect that the solution will always be smooth enough, even at strictly positive time, to give clear meaning to those nonlinear

terms which depend on the first spatial derivatives of $A(t)$, since $A(t)$ need not be in $H_1(\mathbb{R}^3)$ for $t > 0$.

To be more precise, denote by $B := dA + A \wedge A$ the curvature (magnetic field) of the connection form A over \mathbb{R}^3 . Then the Yang-Mills heat equation is

$$\frac{\partial}{\partial t} A(x, t) = -d_A^* B, \quad (1.4)$$

where $d_A^* = d^* +$ the interior product by A . One can verify easily that this has the form indicated in (1.2). Recall that a function $g : \mathbb{R}^3 \rightarrow K$ determines the gauge transformation $A \mapsto A^g$, defined by

$$A^g = g^{-1} A g + g^{-1} dg. \quad (1.5)$$

This in turn induces an action on the curvature given by $B \mapsto g^{-1} B g$. In particular if $A = 0$ then $B = 0$ and the curvature of the “pure gauge” $g^{-1} dg$ is therefore zero. Thus if $A_0 = g^{-1} dg$ then $A(t) := g^{-1} dg$ is clearly a solution to the Yang-Mills heat equation (1.4). It can happen, therefore, that if the initial data is only in $H_{1/2}$, the solution need be no smoother than this for positive time. Yet, once one has computed the curvature in some generalized sense, the equation (1.4) may now involve only classical derivatives of the curvature (which is zero in this example). Thus the first spatial derivatives of $A(t)$ need to be defined in some generalized sense in this example while the needed second derivatives are definable classically. This is the reverse of what one usually considers to be a weak solution. The main theorem of this paper will prove the existence and uniqueness of long time solutions to the equation (1.4) for initial data $A_0 \in H_{1/2}$, wherein the notion of solution will allow first spatial derivatives of $A(t)$ to exist only in a generalized sense while the resulting “weak” curvature of $A(t)$ is actually in H_1 for all $t > 0$. The theorem will also point to the source of this phenomenon by showing that there is a gauge function g_0 such that $A(t)^{g_0}$ is itself in H_1 for all $t > 0$. Otherwise said, any initial data $A_0 \in H_{1/2}$ gives rise to a strong solution up to gauge transformation.

The question of existence and uniqueness of solutions to the Yang-Mills heat equation is of intrinsic interest, partly because it is a naturally occurring quasilinear diffusion equation and partly because of the way that gauge invariance intervenes in the very formulation of the Cauchy problem. But, as in [2, 3], this work is ultimately aimed at the construction of gauge invariant functions of distributional initial data by a completion procedure sketched in the introduction to [2], in an anticipated application to quantum field

theory. In order to construct local observables for this application we will be interested in solutions over a bounded open subset of \mathbb{R}^3 , as well as over all of \mathbb{R}^3 . The question of boundary conditions therefore arises. In [2] we considered Dirichlet, Neumann and Marini boundary conditions. The latter consists in setting the normal component of the curvature to zero on the boundary, [21, 22, 23, 24]. These are the only boundary conditions commensurate with the intended applications to quantum field theory. Solutions satisfying Marini boundary conditions will be derived from solutions satisfying Neumann boundary conditions in a future work. In this paper we will only consider Dirichlet and Neumann boundary conditions. The meaning of the space $H_{1/2}$ will henceforth depend on the choice of boundary conditions.

We are also going to derive existence and uniqueness theorems in case the initial data A_0 is in the Sobolev space H_a for some $a > 1/2$. This will illuminate the way in which some results and some techniques break down as a decreases to its critical value, $1/2$. In particular we will see that the usual contraction mechanism for proving existence of solutions to integral equations breaks down as $a \downarrow 1/2$ and must be replaced by a different contraction mechanism, special to the Yang-Mills heat equation.

The strategy for proving existence of solutions to the initial value problem (1.4), (1.3) consists of the following components.

ZDS procedure. The Zwanziger-Donaldson-Sadun (ZDS) method of ameliorating the failed ellipticity of d^*d will underly the approach in this paper, as it did in [2]. In the ZDS method one deals at first with a modified version of (1.4), obtained by adding a term to the right hand side, which makes the resulting equation strictly parabolic. The so augmented equation is

$$\frac{\partial}{\partial t}C(t) = -d_C^*B_C - d_C d^*C, \quad (1.6)$$

wherein $C(t)$ is a \mathfrak{k} valued 1-form with the same initial data A_0 as (1.3) and $B_C(t)$ is the curvature of $C(t)$. The desired solution $A(t)$ is then recovered from $C(t)$ by a time dependent gauge transformation,

$$A(t) = C(t)^{g(t)}, \quad (1.7)$$

where $g(t, x)$ is determined from $C(\cdot)$ for each point $x \in \mathbb{R}^3$ by a simple ordinary differential equation:

$$\frac{d}{dt}g(t, x) = (d^*C(t, x))g(t, x), \quad g(0, x) = \text{identity element of } K. \quad (1.8)$$

The difficulty in carrying out the ZDS procedure for the recovery of $A(\cdot)$ from $C(\cdot)$ arises from the fact that $d^*C(t)$ has very singular behavior as $t \downarrow 0$. Indeed $d^*C(0)$ need only be in $H_{-1/2}$. This reflects itself in a corresponding degree of irregularity of the gauge function $g(t, \cdot)$ and its spatial differential $g(t)^{-1}dg(t)$, both of which enter into the transformation (1.7). To carry out the ZDS procedure it will be necessary to understand first the nature of the group of gauge functions in which each $g(t)$ lies.

Gauge groups. If $A_0 \in H_{1/2}(\mathbb{R}^3)$ then the solution $C(\cdot)$ to (1.6) will be shown to be a continuous function into $H_{1/2}(\mathbb{R}^3)$. We wish to construct a solution $A(\cdot)$ of (1.4) which is also a continuous function into $H_{1/2}(\mathbb{R}^3)$. A gauge transformation, $C \mapsto g^{-1}Cg + g^{-1}dg$, such as occurs in (1.7), will take $H_{1/2}$ into itself if $g \in H_{3/2}(\mathbb{R}^3, K)$. The statement that $g \in H_{3/2}(\mathbb{R}^3; K)$ needs an interpretation that makes this set of gauge functions into a topological group in a way that serves the needs of the ZDS procedure. It would be reasonable, for example, to define such functions to be those of the form $g(x) = \exp(\alpha(x))$, with $\alpha \in H_{3/2}(\mathbb{R}^3; \mathfrak{k})$. But the $H_{3/2}$ norm of $\alpha(\cdot)$ just barely fails to control the supremum norm of α , with the result that $\exp(\alpha(x))$ wraps around K in an uncontrolled manner as x runs over \mathbb{R}^3 , leading to failure of this set to be a topological group as well as failure to serve the needs of the ZDS procedure. We will define instead a group $\mathcal{G}_{3/2}$ of gauge functions, which in its natural metric topology is a complete topological group in the pointwise product and which serves the needs of the ZDS procedure. Similarly, if $A_0 \in H_a$ for some $a \in (1/2, 1]$ we will define a group \mathcal{G}_{1+a} of gauge functions appropriate for implementing the ZDS procedure in this case. For $a > 1/2$ this group is a nice Hilbert manifold, whereas for $a = 1/2$ there appears to be no tangent space at the identity. This distinction is one of the many ways that distinguish the case $a > 1/2$ from the critical case $a = 1/2$.

As to whether the solution $g(t)$ to (1.8) actually lies in $\mathcal{G}_{3/2}$ for each t depends on the nature of the coefficient $d^*C(t)$, which typically has a strong singularity at $t = 0$, as already noted. Most of this paper, accordingly, is devoted to proving that the function $t \mapsto g(t)$ is actually a continuous function into $\mathcal{G}_{3/2}$ (or into \mathcal{G}_{1+a} if $A_0 \in H_a$). The proof of this, in turn, relies on obtaining detailed information about the singular initial behavior of the solution $C(\cdot)$ of the augmented equation (1.6).

Initial behavior of $C(\cdot)$. The nature of the initial singularity of the solution $C(\cdot)$ which is needed to establish the required properties of the conversion functions $g(t)$ will be analyzed in three steps. First, some knowledge of the singular behavior of $C(t)$ as $t \downarrow 0$ comes immediately from the fact that the

solution lies in the path space that will be used for proving existence of a solution to the integral equation corresponding to (1.6). Second, we will derive energy estimates, which use the fact that the function $C(\cdot)$ not only lies in the path space but is also a solution of the augmented equation (1.6). Generally this gives only L^p information for $2 \leq p \leq 6$ by Gaffney-Friedrichs-Sobolev inequalities. Third, we will derive information from a Neumann domination technique that builds on the previous energy estimates. This will give L^p information for all $p \leq \infty$.

Finite action. The functional

$$\int_0^1 t^{-1/2} \|B(t)\|_{L^2(M)}^2 dt \quad (1.9)$$

is a gauge invariant functional of the initial data A_0 , wherein $M = \mathbb{R}^3$ or is a bounded open subset of \mathbb{R}^3 . It captures in a gauge invariant way the $H_{1/2}$ norm of A_0 , which is itself not a gauge invariant norm. It controls many of the estimates needed in this $H_{1/2}$ theory. It has thus important technical usefulness for us in this paper and important conceptual significance for the applications. We will say that a solution to the Yang-Mills heat equation has finite action if the functional (1.9) is finite. (This terminology is motivated by the fact that, when (1.9) is finite, the initial data A_0 has an extension to a time interval in Minkowski space which assigns finite value to the magnetic contribution to the Lagrangian.) We will prove that if $\|A_0\|_{H_{1/2}}$ is sufficiently small, then the solution has finite action. If $\|A_0\|_{H_a} < \infty$ for some $a > 1/2$ then the corresponding “a-action” is always finite. This is yet another distinction between the critical case $a = 1/2$ and the cases $a > 1/2$.

The techniques in this paper rely heavily on the results in [2] and [3], which deal with the Yang-Mills heat equation for initial data in H_1 . The Bianchi identity, $d_A B = 0$, encodes much of what is special about the form of the Yang-Mills heat equation. To take advantage of this it is essential to formulate identities and inequalities in terms of the gauge covariant exterior derivative d_A and its adjoint. All of the key inequalities we get with this use are gauge invariant. The gauge invariant Gaffney-Friedrichs inequality, which gives information about the gradient of a form in terms of the exterior derivative and co-derivative of the form, is needed here, as in [2], for enabling use of Sobolev inequalities. It will continue to be a major tool.

Once one has propagated the initial data for a short time one can apply the results of [2] to establish long time existence of the solution $A(\cdot)$. Concerning uniqueness of solutions, a standard kind of proof, based on a

Gronwall type of argument, applies to our equation in case $a > 1/2$, but breaks down when $a = 1/2$. For $a = 1/2$ we will give a proof of uniqueness which is special to the structures at our disposal.

The first proof of existence of solutions of the Yang-Mills heat equation over a three dimensional closed manifold with H_1 initial data was given by J. Råde in [32]. In his proof, Råde took the magnetic field as an independent unknown function, in addition to the gauge potential A , and showed in the end that, for the joint solution of the resulting parabolic system, the independent magnetic field is indeed the curvature of A . This method goes back to Ginibre and Velo [13, 14] in the context of the hyperbolic Yang-Mills equations in $2 + 1$ space time dimensions and to DeTurck [7] in the context of the parabolic Ricci flow problem. The ZDS procedure used in the present paper produces the gauge function g_0 for which $A(t)^{g_0}$ is a strong solution. Any method alternative to the ZDS procedure would also be required to produce such a gauge function because of the conceptual role that g_0 plays when the initial data A_0 is in H_a with $a < 1$. Remark 2.16 discusses this further. More history of the Yang-Mills heat flow is given in the introduction of [2].

The ZDS procedure, which we used in [2, 3] and in the present paper, has its origins in a suggestion by D. Zwanziger [41] in the context of stochastic quantization, and in the work of Donaldson [8] and Sadun [35]. Recently, Sung-Jin Oh [29, 30] has used the Yang-Mills heat equation to provide a very novel way to attack the Cauchy problem for the hyperbolic Yang-Mills equation in $3 + 1$ space time dimensions. He combines the hyperbolic and parabolic equations into one system in order to force a continuously changing gauge choice, which is favorable for the hyperbolic system. The ZDS method underlies his analysis of the parabolic portion of the system. He is concerned with H_1 initial data for the heat equation since it matches with the initial data of the hyperbolic equation. But the critical Sobolev index for the hyperbolic equation is also $1/2$. One could reasonably anticipate that Oh's method might be implementable for $H_{1/2}$ initial data for the hyperbolic equation by combining it with the heat equation results which we derive here.

Complementary to the question of long time existence of the Yang-Mills heat flow is the question of short time blow up of solutions. In four or more dimensions solutions tend to blow up in a finite time unless restricted by some strong symmetry conditions or some additional invariant of the equation. See e.g. [37, 36, 20] and [8]. But in [18], J. Grotowski showed that over \mathbb{R}^n , with $n \geq 5$, solutions with smooth initial data can blow up in a finite time even if the initial data and solution are restricted by stringent symmetry

conditions. In [28] and [40] the nature of the singularity formation has been investigated and [19] makes a comparison of blow up phenomena in Yang-Mills evolution and harmonic map evolution. In [11, 12] it was shown how singularity formation is associated with non-uniqueness of the flow. Semi-probabilistic methods for determining blow up and no blow up are described in [1] and [31].

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2 Statement of results

M will denote either \mathbb{R}^3 or the closure of a bounded, open set in \mathbb{R}^3 with smooth boundary. K will denote a compact Lie group contained in the unitary (resp. orthogonal) group of some finite dimensional inner product space \mathcal{V} . $\langle \cdot, \cdot \rangle$ will denote an $\text{Ad } K$ invariant inner product on its Lie algebra \mathfrak{k} .

We continue to use the commutator-wedge product notation from [2] given by $[(\xi dx^{j_1} \wedge \cdots \wedge dx^{j_p}) \wedge (\eta dx^{k_1} \wedge \cdots \wedge dx^{k_q})] = [\xi, \eta] dx^{j_1} \wedge \cdots \wedge dx^{k_q}$ for \mathfrak{k} valued functions ξ and η . For a \mathfrak{k} valued connection form A on M the exterior derivative d and the gauge invariant exterior derivative d_A are related by $d_A \omega = d\omega + [A \wedge \omega]$, wherein ω is a \mathfrak{k} valued p-form. If u is a \mathfrak{k} valued r-form and v is a \mathfrak{k} valued p-form with $r \leq p$ then the interior product $[u \lrcorner v]$ is the (p-r)-form defined by $\langle [u \lrcorner v], \omega \rangle_{\Lambda^{p-r} \otimes \mathfrak{k}} = \langle v, [u \wedge \omega] \rangle_{\Lambda^p \otimes \mathfrak{k}}$ for all (p-r)-forms ω . The adjoint of d_A in $L^2(M; \Lambda^* \otimes \mathfrak{k})$ is then given by $d_A^* \omega = d^* \omega + [A \lrcorner \omega]$ for any \mathfrak{k} valued p-form ω . In the following, W_1 refers to the Sobolev space of order one without boundary conditions.

2.1 Strong solutions

Definition 2.1 Let $0 < T \leq \infty$. A *strong solution* to the Yang-Mills heat equation over $[0, T)$ is a continuous function

$$A(\cdot) : [0, T) \rightarrow L^2(M; \Lambda^1 \otimes \mathfrak{k}) \subset \{\mathfrak{k}\text{-valued 1-forms on } M\} \quad (2.1)$$

such that

$$a) A(t) \in W_1 \text{ for all } t \in (0, T) \text{ and } A(\cdot) : (0, T) \rightarrow W_1 \text{ is continuous,} \quad (2.2)$$

$$b) B(t) := dA(t) + A(t) \wedge A(t) \in W_1 \text{ for each } t \in (0, T), \quad (2.3)$$

$$c) \text{ the strong } L^2(M) \text{ derivative } A'(t) \equiv dA(t)/dt \text{ exists on } (0, T), \text{ and}$$

$$A'(\cdot) : (0, T) \rightarrow L^2(M) \text{ is continuous,} \quad (2.4)$$

$$d) A'(t) = -d_{A(t)}^* B(t) \text{ for each } t \in (0, T). \quad (2.5)$$

An *almost strong solution* is a function $A(\cdot)$ satisfying all of the preceding conditions except a). In this case the spatial exterior derivative $dA(t)$, which enters into the definition of the curvature B , must be interpreted as a weak derivative.

Remark 2.2 (Weak curvature) If a) does not hold then the weak derivatives $\partial A(t)/\partial x_j$, $j = 1, 2, 3$ need not be functions. Yet condition b) requires that the particular combination of derivatives that enter into $dA(t)$ be a function. This can happen easily, as one sees in the example $A = d\lambda$, where λ is an arbitrary real valued distribution on \mathbb{R}^3 . In this case the distribution dA is the function identically equal to zero. Many of the problems that we will need to deal with in this paper arise from the presence of pure gauges, which are the non-commutative analogs of this example.

Definition 2.3 (Boundary conditions) Let

$$-\Delta = d^*d + dd^* \text{ on } \mathfrak{k} \text{ valued 1-forms over } M \quad (2.6)$$

In case $M \neq \mathbb{R}^3$ then, for a \mathfrak{k} valued 1-form ω on M , the Neumann and Dirichlet domains for Δ are given by

$$(N) \quad \omega_{norm} = 0, \quad d\omega_{norm} = 0 \quad \text{Neumann domain,} \quad (2.7)$$

$$(D) \quad \omega_{tan} = 0, \quad (d^*\omega)|_{\partial M} = 0 \quad \text{Dirichlet domain,} \quad (2.8)$$

where ω_{norm} and ω_{tan} are the normal and tangential components of ω on ∂M . The boundary conditions (N) and (D) are respectively absolute and relative boundary conditions in the sense of Ray and Singer, [33]. See [2, Remark 2.11] for further discussion. Both versions of $-\Delta$ are non-negative self-adjoint operators on the appropriate domains. The corresponding Sobolev spaces are

$$H_a = \text{Domain of } (-\Delta)^{a/2} \text{ in } L^2(M; \Lambda^1 \otimes \mathfrak{k}) \quad (2.9)$$

with norm

$$\|\omega\|_{H_a} = \|(1 - \Delta)^{a/2} \omega\|_{L^2(M; \Lambda^1 \otimes \mathfrak{k})}. \quad (2.10)$$

With this definition one has

$$\|\omega\|_{H_a} \leq c_{a,b} \|\omega\|_{H_b} \quad \text{if } 0 \leq a \leq b < \infty, \quad (2.11)$$

where $c_{a,b}$ is a constant independent of M . In particular, $H_b \subset H_a$ when $a \leq b$.

A theorem stated without specifying Neumann or Dirichlet boundary conditions applies to both cases when $M \neq \mathbb{R}^3$, as well as to the case $M = \mathbb{R}^3$.

Remark 2.4 (More about boundary conditions) Suppose that M is the closure of a bounded open set in \mathbb{R}^3 with smooth boundary. It is well understood that the usual Neumann Laplacian for real valued functions u on M can be simply defined as the unique self-adjoint operator $-\Delta_N$ whose quadratic form is given by $Q_N(u) = \int_M |\nabla u(x)|^2 dx$. $\mathcal{D}(Q_N)$ consists of all measurable functions u for which $Q_N(u)$ is finite. Δ_N is unique under the usual assumption that $\mathcal{D}(\Delta_N) \subset \mathcal{D}(Q_N)$. Similarly the Dirichlet Laplacian is the unique self-adjoint operator $-\Delta_D$ whose quadratic form Q_D is again given by this integral but with domain consisting of those functions u in the domain of Q_N which are zero on ∂M . (More precisely Q_D is the closure of the form $Q_N|_{C_c^\infty(M^{int})}$). In this paper we are concerned with Laplacians on 1-forms over M . There are two natural senses for a 1-form ω to be zero on the boundary. Define

$$Q_{norm}(\omega) = \int_M \left(|d\omega(x)|_{\Lambda^2}^2 + |\delta\omega(x)|_{\Lambda^0}^2 \right) dx, \quad \omega_{norm} = 0 \text{ on } \partial M \quad (2.12)$$

$$Q_{tan}(\omega) = \int_M \left(|d\omega(x)|_{\Lambda^2}^2 + |\delta\omega(x)|_{\Lambda^0}^2 \right) dx, \quad \omega_{tan} = 0 \text{ on } \partial M, \quad (2.13)$$

where d denotes the exterior derivative on 1-forms in $C^\infty(M)$ and δ denotes the coderivative on 1-forms in $C^\infty(M)$. The domains of both of these quadratic forms is specified by imposing a Dirichlet type condition on ω . Yet the Laplacian naturally associated to each one forces a form ω in its domain to satisfy not only the Dirichlet type condition $\omega_{norm} = 0$, resp. $\omega_{tan} = 0$, but also a derivative type condition $(d\omega)_{norm} = 0$, resp. $(\delta\omega)_{tan} = 0$. The latter are Neumann type conditions. Thus the two Laplacians associated to the two quadratic forms are neither Dirichlet Laplacians nor Neumann

Laplacians. As noted above, Ray and Singer [33] have called the associated boundary conditions absolute and relative boundary conditions, respectively, because of their role in absolute and relative cohomology. We are going to continue to follow Conner [5], who refers to them as the Neumann Laplacian and Dirichlet Laplacian, respectively, because they reduce to these on zero forms. We will often deal with small fractional powers of the Laplacian. When $1/2 < a < 3/2$ the domain of $(-\Delta_N)^{a/2}$ is restricted only by the boundary condition $\omega_{norm} = 0$ and not by a condition on the derivative of ω . A similar comment applies to $(-\Delta_D)^{a/2}$ and $\omega_{tan} = 0$.

2.2 Gauge groups

Notation 2.5 (Gauge groups) There are several gauge groups over M that mediate the formulation of the existence theorems. A measurable function $g : M \rightarrow K \subset \text{End } \mathcal{V}$ is a bounded function into a linear space and therefore its weak derivatives over the interior of M are well defined. We will be interested in such functions g for which the weak derivatives $\partial_j g$ are in $L^2(M; \text{End } \mathcal{V})$, $j = 1, 2, 3$. We will say that $g \in W_1$ in this case. For a function $g \in W_1$ the functions $x \mapsto g(x)^{-1} \partial_j g(x)$ take their values a.e. in the Lie algebra $\mathfrak{k} \subset \text{End } \mathcal{V}$. Thus the \mathfrak{k} valued 1-form $g^{-1} dg$ lies in $L^2(M; \Lambda^1 \otimes \mathfrak{k})$. Postponing till Section 5 a precise discussion of the boundary conditions on g itself, let us define \mathcal{G}_{1+a} to consist of those functions $g \in W_1$ such that $\|g^{-1} dg\|_{H_a} < \infty$.

Theorem 2.6 *The set \mathcal{G}_{1+a} is a complete topological group under the pointwise product if $1/2 \leq a \leq 1$. (See Section 5 for the topology.)*

Remark 2.7 (Failure of $\exp(H_{3/2})$) As noted in the introduction, there is a technical disadvantage in attempting to use the representation $g(x) = \exp((\alpha(x)))$ for elements of the critical gauge group $\mathcal{G}_{3/2}$. The inverse of the exponential map can be poorly behaved at some points $\exp(\alpha(x))$ because $\exp(\alpha(x))$ can wrap around the whole group K even when $\|\alpha\|_{H_{3/2}}$ is small, leading to failure of inversion and multiplication to be continuous. In four dimensions the analogous borderline gauge group is \mathcal{G}_2 , since the Sobolev W_2 norm also just fails to control the supremum norm. K. Uhlenbeck has already pointed out [39, page 33] that in this four dimensional case multiplication and inversion fail to be continuous in $\mathcal{G}_2(\text{Ball in } \mathbb{R}^4)$ if one defines the Sobolev behavior of an element of \mathcal{G}_2 by means of the exponential representation. In our definition of the topological group $\mathcal{G}_{3/2}$, the set $\{\exp((\alpha(x))) : \alpha \in H_{3/2}(M)\}$

is not likely even to cover any neighborhood of the identity. Further discussion of this can be found in Remark 5.21.

2.3 Main theorem

Definition 2.8 (Finite a -action) An almost strong solution $A(\cdot)$ to the Yang-Mills heat equation has *finite a -action* if

$$\int_0^\epsilon s^{-a} \|B(s)\|_{L^2(M)}^2 ds < \infty \quad \text{for some } \epsilon > 0, \quad (2.14)$$

where $B(s)$ is the curvature of $A(s)$. This definition is of interest for $1/2 \leq a < 1$. For $a = 1/2$ it reduces to the definition (1.9) of finite action given in the introduction.

Definition 2.9 (Convexity of M) For some of the results in this paper we will need to assume that M , if it isn't all of \mathbb{R}^3 , is the closure of a bounded, open convex subset of \mathbb{R}^3 with smooth boundary. This ensures that the second fundamental form of ∂M is everywhere non-negative. When needed, we will simply refer to such a set as convex. Discussion of when convexity of M is not needed may be found in Remark 2.17.

Theorem 2.10 ($a > 1/2$) *Let $1/2 < a < 1$. Assume that M is all of \mathbb{R}^3 or is convex in the sense of Definition 2.9. Suppose that $A_0 \in H_a(M)$. Then*

1) *there exists an almost strong solution $A(t)$ to (2.5) over $[0, \infty)$ with initial value A_0 and with the following properties.*

2) *There exists a gauge function $g_0 \in \mathcal{G}_{1+a}$ such that $A(t)^{g_0}$ is a strong solution.*

3) *$A(\cdot)$ and $A(\cdot)^{g_0}$ are continuous functions on $[0, \infty)$ into H_a .*

4) *$A(\cdot)$ and $A(\cdot)^{g_0}$ both have finite a -action.*

5) *If $M \neq \mathbb{R}^3$ then the following boundary condition is satisfied by both $A(t)$ and $A(t)^{g_0}$.*

$$(curvature|_t)_{norm} = 0 \quad \text{in case } (N) \quad \forall t > 0 \quad (2.15)$$

$$(curvature|_t)_{tan} = 0 \quad \text{in case } (D) \quad \forall t > 0. \quad (2.16)$$

Moreover

$$(A(t)^{g_0})_{norm} = 0 \quad \text{in case } (N) \quad \forall t > 0 \quad (2.17)$$

$$(A(t)^{g_0})_{tan} = 0 \quad \text{in case } (D) \quad \forall t > 0. \quad (2.18)$$

6) Strong solutions are unique among solutions with finite a -action under the boundary condition (in case $M \neq \mathbb{R}^3$)

$$B(t)_{norm} = 0 \text{ for all } t > 0 \text{ in case (N)} \quad (2.19)$$

$$A(t)_{tan} = 0 \text{ for all } t > 0 \text{ in case (D)}. \quad (2.20)$$

Theorem 2.11 ($a = 1/2$). Assume that M is all of \mathbb{R}^3 or is convex in the sense of Definition 2.9. Suppose that $A_0 \in H_{1/2}(M)$. Then

1) there exists an almost strong solution $A(t)$ to (2.5) over $[0, \infty)$ with initial value A_0 . Its curvature satisfies the boundary conditions (2.15) resp. (2.16) if $M \neq \mathbb{R}^3$.

2) There exists a gauge function g_0 such that $A(t)^{g_0}$ is a strong solution. $A(t)^{g_0}$ satisfies the boundary conditions (2.15) resp. (2.16) as well as (2.17) resp. (2.18) when $M \neq \mathbb{R}^3$.

3) If $\|A_0\|_{H_{1/2}}$ is sufficiently small then $A(\cdot)$ and $A(\cdot)^{g_0}$ have finite $(1/2)$ -action. In this case one may choose g_0 to lie in $\mathcal{G}_{3/2}$

4) If $\|A_0\|_{H_{1/2}}$ is sufficiently small then $A(\cdot)$ is a continuous function from $[0, \infty)$ into $H_{1/2}$. If, in addition, g_0 is chosen to lie in $\mathcal{G}_{3/2}$ then $A^{g_0}(\cdot) : [0, \infty) \rightarrow H_{1/2}$ is also continuous.

5) Strong solutions are unique among solutions with finite $(1/2)$ -action under the boundary condition (in case $M \neq \mathbb{R}^3$) (2.19) resp. (2.20).

Remark 2.12 (Meaning of boundary conditions) Since the curvature $B(t)$ is in W_1 for $t > 0$, by the definition of a strong or almost strong solution, the restriction $B(t)|_{\partial M}$ is well defined a.e. on ∂M , and consequently the boundary conditions (2.15) and (2.16) are meaningful. However $A(t)$ need not be in W_1 for an almost strong solution. The boundary conditions (2.17) and (2.18) may therefore not be meaningful for $A(t)$ itself but only for the gauge transformed solution $A(t)^{g_0}$. This relates to Remarks 2.2 and 2.13.

Remark 2.13 (Uniqueness for almost strong solutions) Our proof of uniqueness depends on initial behavior bounds for $\|B(t)\|_\infty$, which we will derive via Neumann domination techniques in Section 4.7. These bounds depend in turn on energy bounds for initial behavior, which in turn depend on finite a -action. For this reason our formulation of uniqueness specifies uniqueness only among solutions of finite action. It is reasonable to ask whether uniqueness holds among almost strong solutions of finite action. In the case that $M \neq \mathbb{R}^3$ one needs of course to impose a boundary condition such as (2.19) or

(2.20). Since $B(t) \in W_1$ for almost strong solutions, the requirement (2.19) is meaningful for an almost strong solution for Neumann boundary conditions. But, since $A(t)$ need not be in $W_1(M)$ for an almost strong solution, one would need to interpret the boundary condition (2.20) properly to address uniqueness for almost strong solutions in the case of Dirichlet boundary conditions. In this case the assertion that $A(t) \in H_a$ already reflects a boundary condition on $A(t)$. For example, it can be expected, on the basis of Fujiwara's theorem [10], that $A(t)_{tan} = 0$ when $a > 1/2$ and that this holds in a mean sense when $a = 1/2$.

Aside from the problem of formulation of uniqueness for almost strong solutions (in Dirichlet case) there are some (seemingly) technical issues in justifying the computations that lead to the key inequality (7.53) needed for the proof of uniqueness. We do not have available for almost strong solutions such a good approximation mechanism as can be found in [2, Lemma 9.1]. The issues raised by the question of uniqueness for almost strong solutions of finite action relate to other problems, which will be addressed elsewhere. We will not consider uniqueness of almost strong solutions in this paper.

Remark 2.14 (Smoothness) The solution A^{g_0} produced in Theorems 2.10 and 2.11 is actually in $C^\infty((0, T] \times M; \Lambda^1 \otimes \mathfrak{k})$ for some time $T < \infty$. Very likely it is in $C^\infty((0, \infty) \times M; \Lambda^1 \otimes \mathfrak{k})$. But our proof does not rule out the possibility that it loses smoothness if one doesn't make occasional gauge transforms, even though it remains in $H_1(M)$ for all $t > 0$. See Theorem 7.1. However it will be shown in [4] that gauge covariant derivatives of all orders exist.

Remark 2.15 The gauge transformation g_0 that converts an almost strong solution, $A(\cdot)$, to a strong solution $A(\cdot)^{g_0}$ is not unique: If g_1 lies in the gauge group \mathcal{G}_2 then it gauge transforms a strong solution to another strong solution and consequently $A(\cdot)^{g_0 g_1}$ is also a strong solution. It will be shown in [16] that this is the extent of the non-uniqueness. Denote by $\mathcal{Y}_a(M)$ the set of strong solutions over M with initial data in $H_a(M)$. (Choose either Dirichlet or Neumann boundary conditions when $M \neq \mathbb{R}^3$.) $\mathcal{G}_{1+a}(M)$ acts continuously on $H_a(M)$ in its natural action $A \mapsto g^{-1}Ag + g^{-1}dg$. Theorem 2.10 asserts that each fiber in the bundle $H_a(M) \mapsto H_a(M)/\mathcal{G}_{1+a}(M)$ contains at least one element of $\mathcal{Y}_a(M)$. Thus we have

$$H_a/\mathcal{G}_{1+a} = \mathcal{Y}_a/\mathcal{G}_2, \quad (2.21)$$

given the assertion above concerning the extent of the non-uniqueness of g_0 . We will see in [16] that \mathcal{Y}_a is a complete Riemannian manifold with respect to a \mathcal{G}_2 invariant Riemannian metric associated to the action (2.14). An identity similar to (2.21) holds also for $a = 1/2$ if one restricts to small $\|A_0\|_{H_{1/2}}$ in accordance with Theorem 2.11. In the sense of (2.21), the non-gauge invariant norm on the linear space $H_a(M)$ is captured, up to gauge transformations, by a gauge invariant Riemannian metric on the manifold \mathcal{Y}_a .

Remark 2.16 If the initial data A_0 is in H_1 then the role of the gauge functions $g(t)$ produced by the ZDS procedure, discussed in the introduction and in the next subsection, is an auxiliary one in the sense that it is needed only to produce the solution A from C . But if A_0 is only in H_a for some $a < 1$ then these gauge functions play a more fundamental role. A solution with initial value $A_0 \in H_a$ need not be a strong solution. But the ZDS procedure produces a gauge function g_0 which transforms A_0 into another element of H_a , which is the initial value of a strong solution. This is a reflection of the identity (2.21). The gauge function g_0 therefore plays an indispensable role in the formulation of the Cauchy problem. There does not appear to be a way to decompose the initial data space H_a into “longitudinal plus transverse” subsets that propagate as “constant solutions”, respectively strong solutions, so that the transverse subset might serve as a section over the quotient space H_a/\mathcal{G}_{1+a} .

Remark 2.17 (Convexity of M) The convexity of M enters only in the proof of Neumann domination bounds. Convexity of M is therefore not needed in our discussion of gauge groups (Section 5) or in our proof of existence and uniqueness of solutions to the augmented Yang-Mills heat equation (Section 3). It is reasonable to anticipate that convexity could be replaced by a negative lower bound on the second fundamental form of ∂M in some version of the Neumann domination bounds.

Remark 2.18 (Boundedness of M) In case $M \neq \mathbb{R}^3$ we have assumed that M is bounded, in addition to being the closure of an open set with smooth boundary. Together, these assumptions ensure that standard Sobolev inequalities hold over M as well as the needed operator bounds for the Neumann Laplacian heat semigroup. But boundedness of M is not essential for these to hold. Classes of unbounded domains for which these hold have been

extensively investigated. For the anticipated applications of this paper, it suffices to note that these unbounded domains include half-spaces and infinite slabs. If M is unbounded and the standard Sobolev inequalities and Neumann heat operator bounds hold, then those of our results which are dependent on the Gaffney-Friedrichs inequality will also hold when, in addition, the second fundamental form of ∂M is bounded below, and those of our results dependent on Neumann domination will also hold when, in addition, the second fundamental form of ∂M is non-negative. Of course, if $M = \mathbb{R}^3$ then the Neumann Laplacian is to be replaced by the self-adjoint version of the Laplacian over \mathbb{R}^3 . All of our results hold in this case.

2.4 The ZDS procedure and the augmented equation

The proof of Theorems 2.10 and 2.11 will be based on the ZDS procedure, already discussed in the introduction, and in particular on use of the following modified Yang-Mills heat equation. See [2] for more discussion of the ZDS procedure.

Definition 2.19 (Augmented equation.) The *augmented Yang-Mills heat equation* is

$$-\frac{\partial}{\partial t}C(t) = d_{C(t)}^*B_C(t) + d_{C(t)}d^*C(t), \quad C(0) = C_0. \quad (2.22)$$

Here $C(t)$ is a \mathfrak{k} valued 1-form on M for each $t \geq 0$ and $B_C(t)$ is its curvature. Equation (2.22) differs from the Yang-Mills heat equation (2.5) by the addition of the second term on the right. The added term makes the equation strictly parabolic. If $M \neq \mathbb{R}^3$ the equation goes along with one of the following two kinds of boundary conditions, (N) (for Neumann) or (D) (for Dirichlet).

$$(N) \quad C(t)_{norm} = 0 \text{ for } t \geq 0, \quad (B_C(t))_{norm} = 0 \text{ for } t > 0 \quad (2.23)$$

$$(D) \quad C(t)_{tan} = 0 \text{ for } t \geq 0, \quad (d^*C(t))|_{\partial M} = 0 \text{ for } t > 0. \quad (2.24)$$

By a *strong solution* to the augmented Yang-Mills heat equation (2.22) over an interval $[0, T]$ we mean a continuous function $C(\cdot) : [0, T] \rightarrow L^2(M; \Lambda^1 \otimes \mathfrak{k})$ satisfying the four conditions a) - d) of Definition 2.1, with A replaced by C , B replaced by B_C and (2.5) replaced by (2.22). We will be concerned only with $T < \infty$ for the augmented equation.

Theorem 2.20 (*Solutions to the augmented equation.*) Let $1/2 \leq a < 1$. Suppose that $C_0 \in H_a(M)$. Then there exists a real number $T > 0$ and a continuous function $C : [0, T] \rightarrow H_a(M)$ such that $C(0) = C_0$ and

a) $C(\cdot)$ is a strong solution to the augmented equation (2.22) over $(0, T]$ satisfying the respective boundary conditions (2.23) or (2.24), when $M \neq \mathbb{R}^3$.

b) $t^{(1-a)} \|C(t)\|_{H_1}^2 \rightarrow 0$ as $t \downarrow 0$.

The solution is unique under the preceding conditions. Moreover $C(\cdot)$ lies in $C^\infty((0, T) \times M; \Lambda^1 \otimes \mathfrak{k})$

If $a > 1/2$ then the solution has **finite strong a-action** in the sense that

$$\int_0^T s^{-a} \|C(s)\|_{H_1}^2 ds < \infty. \quad (2.25)$$

If $a = 1/2$ and $\|C_0\|_{H_{1/2}}$ is sufficiently small then (2.25) holds with $a = 1/2$.

The proof of this theorem will be given in Section 3.

Remark 2.21 The link between the augmented ymh equation (2.22) and the ymh equation (2.5) is provided by the ZDS procedure outlined in the introduction. As already noted there, much of this paper is concerned with determining the behavior of the initial singularity of $C(\cdot)$ and its derivatives in order to establish the required differentiability properties of the solution $g(\cdot)$ to the Equation (1.8). These, in turn, will give the differentiability properties of $A(\cdot)$ asserted in Theorems 2.10 and 2.11. Thus, most of this paper is devoted to proving the following theorem, which is stated here just for $a = 1/2$ for simplicity.

Theorem 2.22 Assume that $M = \mathbb{R}^3$ or is convex in the sense of Definition 2.9. Suppose that $A_0 \in H_{1/2}$ and that $C(\cdot)$ is a strong solution to the augmented equation (2.22) with finite strong action over $[0, T]$ and with $C_0 = A_0$. Then there exists a continuous function

$$g : [0, T] \rightarrow \mathcal{G}_{3/2} \quad (2.26)$$

such that the gauge transform $A(\cdot)$ defined by

$$A(t) = C(t)^{g(t)} \quad 0 \leq t \leq T \quad (2.27)$$

is an almost strong solution to the Yang-Mills heat equation over $(0, T)$, whose curvature satisfies the boundary condition (2.15) resp. (2.16). The function

$$A(\cdot) : [0, T] \rightarrow H_{1/2} \quad (2.28)$$

is continuous. In particular, $A(t)$ converges in $H_{1/2}$ norm to A_0 as $t \downarrow 0$.

If $0 < \tau < T$ and $g_0 \equiv g(\tau)^{-1}$ then the function $t \mapsto A(t)^{g_0}$ is a strong solution to the Yang-Mills heat equation satisfying, if $M \neq \mathbb{R}^3$, the boundary condition (2.15) resp (2.16) as well as the boundary condition (2.17) resp. (2.18). $A^{g_0}(\cdot)$ is a continuous function on $[0, T]$ into $H_{1/2}$, and in particular $A(t)^{g_0}$ converges in $H_{1/2}$ norm to $A_0^{g_0}$ as $t \downarrow 0$.

Furthermore $A(\cdot)$ and $A^{g_0}(\cdot)$ have finite action:

$$\int_0^T s^{-1/2} \|B(s)\|_2^2 ds < \infty. \quad (2.29)$$

This theorem, along with its analog for $a > 1/2$, will be proved in Section 7, (Theorem 7.1). In case $A_0 \in H_{1/2}$ but $C(\cdot)$ does not have finite strong action one needs a weaker version of Theorem 2.22, based on infinite $(1/2)$ -action, in order to prove items 1) and 2) in Theorem 2.11. The infinite action version of Theorem 2.22 will be stated and proved in Section 7 also (Theorem 7.2).

Remark 2.23 (Contrast with H_1 initial data) In case the initial data A_0 is in $H_1(M)$ there is no need to invoke the use of gauge transformations in the the formulation of the existence theorem because the solution that the ZDS procedure produces is already a strong solution. Here is a version of the main result from [2], formulated in \mathbb{R}^3 rather than over a compact Riemannian manifold.

Theorem 2.24 [2] *Let M be either \mathbb{R}^3 or the closure of a bounded, convex, open subset of \mathbb{R}^3 with smooth boundary. Suppose that $A_0 \in H_1(M)$. Then there exists a strong solution $A(\cdot)$ to the Yang-Mills heat equation (2.5) over $[0, \infty)$ with initial value A_0 . Moreover $A : [0, \infty) \rightarrow H_1$ is continuous.*

Remark 2.25 1. The conclusion of Theorem 2.24 should be contrasted with the conclusions of Theorems 2.10 and 2.11. One does not need to gauge transform the solution to obtain a strong solution when $A_0 \in H_1$.

2. When $M \neq \mathbb{R}^3$, boundary conditions on A_0 are implied in Theorem 2.24 by the assumption that $A_0 \in H_1$. These are $(A_0)_{\text{norm}} = 0$ in case (N)

or $(A_0)_{tan} = 0$ in case (D). Compare Remark 2.4. Moreover the solution satisfies, for all $t > 0$, the boundary conditions (2.15) resp. (2.16) on its curvature, as well as $A(t)_{norm} = 0$ resp. $A(t)_{tan} = 0$. No gauge transform need intervene as in (2.17) and (2.18). Uniqueness holds just under the condition (2.19) resp. (2.20).

3. The case $M = \mathbb{R}^3$ is not stated in [2]. However if $M = \mathbb{R}^3$, then all the steps in the proof in [2] go through without essential change. In fact the proof in this case is considerably simpler because all of the desiderata concerning boundary conditions can be ignored. Finite volume of M is never used.

3 Solutions for the augmented Yang-Mills heat equation

3.1 The integral equation and path space

Throughout Section 3 M will be assumed to be either all of \mathbb{R}^3 or else the closure of a bounded open set in \mathbb{R}^3 with smooth boundary. M need not be convex.

We will convert the augmented Yang-Mills heat equation (2.22) to an integral equation and then show that the integral equation has a unique solution for a short time. In Section 3.6 it will be shown that the solution is actually a strong solution to (2.22).

To carry out the conversion to an integral equation one needs to separate the linear terms from the non-linear terms in (2.22). Throughout this section we will write d for the exterior derivative with the understanding that this represents the maximal or minimal version, in agreement with the boundary conditions when $M \neq \mathbb{R}^3$. See [2] for a discussion of these domains. It will be convenient to use the exterior and interior commutator products of \mathfrak{k} valued forms defined at the beginning of Section 2.

Writing $B \equiv B_C = dC + (1/2)[C \wedge C]$, one can compute that

$$d_C^* B + d_C d^* C = (d^* d + d d^*) C - X(C), \quad (3.1)$$

where X is the first order nonlinear differential operator on \mathfrak{k} valued 1-forms C defined by

$$-X(C) = -[C \lrcorner B] + (1/2)d^*[C \wedge C] + [C, d^* C], \quad C : M \rightarrow \Lambda^1 \otimes \mathfrak{k}. \quad (3.2)$$

The terms in $X(C)$ which are cubic in C involve no derivatives of C while the terms which are quadratic all involve a factor of one spatial derivative of C . As in [2] we will write this symbolically as

$$X(C) = C^3 + C \cdot \partial C. \quad (3.3)$$

$X(C)$ contains all the non-linear terms in Eq. (2.22), which can now be rewritten as

$$C'(t) = \Delta C(t) + X(C(t)), \quad C(0) = C_0, \quad (3.4)$$

wherein Δ is the Neumann or Dirichlet Laplacian defined in Definition 2.3.

Informally, the equation (3.4) is equivalent to the integral equation

$$C(t) = e^{t\Delta}C_0 + \int_0^t e^{(t-\sigma)\Delta}X(C(\sigma))d\sigma. \quad (3.5)$$

A solution to the integral equation (3.5) is sometimes referred to as a mild solution to the differential equation (3.4) [34, Definition 11.15]. We will show in Section 3.6 that such a mild solution is actually a strong solution. The existing general theorems showing that a mild solution is a strong solution seem inapplicable to our case.

Remark 3.1 Any choice of path space within which one wishes to seek a solution to (3.5) with initial data $C_0 \in H_a$ should be contained in $C([0, T]; H_a(M))$ and should include paths having arbitrary initial value in $H_a(M)$. But it must also have a strong enough metric to allow control of the non-linear function $X(C)$. The following path space seems well adapted to this purpose for our particular non-linearities and initial conditions.

Notation 3.2 (Path space.) Suppose that $0 < a < 1$ and $0 < T < \infty$. Let $C_0 \in H_a \equiv H_a(M; \Lambda^1 \otimes \mathfrak{k})$. Define

$$\mathcal{P}_T^a = \left\{ C(\cdot) \in C([0, T]; H_a) \cap C((0, T]; H_1) : \right. \\ \left. \begin{array}{ll} i. & C(0) = C_0 \\ ii. & t^{1-a} \|C(t)\|_{H_1}^2 \rightarrow 0 \text{ as } t \downarrow 0 \end{array} \right\}. \quad (3.6)$$

$$ii. \quad t^{1-a} \|C(t)\|_{H_1}^2 \rightarrow 0 \text{ as } t \downarrow 0 \}. \quad (3.7)$$

Define also

$$|C|_t = \sup_{0 < s \leq t} s^{(1-a)/2} \|C(s)\|_{H_1}, \quad 0 < t \leq T. \quad (3.8)$$

Then, for $C \in \mathcal{P}_T^a$, we have

$$\|C(s)\|_{H_1} \leq s^{(a-1)/2} |C|_t \quad \text{for } 0 < s \leq t \leq T \quad \text{and} \quad (3.9)$$

$$|C|_t \leq |C|_T \quad \text{for } 0 < t \leq T. \quad (3.10)$$

Condition *ii.* ensures that

$$|C|_t \rightarrow 0, \text{ as } t \downarrow 0. \quad (3.11)$$

\mathcal{P}_T^a is a complete metric space in the metric

$$\text{dist}(C_1, C_2) = \sup_{0 \leq t \leq T} \|C_1(t) - C_2(t)\|_{H_a} + |C_1 - C_2|_T. \quad (3.12)$$

The inequality (3.9) ensures that, for some Sobolev constant κ_6 , one has

$$\|C(s)\|_6 \leq s^{(a-1)/2} |C|_t \kappa_6, \quad \text{for } 0 < s \leq t. \quad (3.13)$$

These spaces will be useful only for $1/2 \leq a < 1$.

The next theorem is the mild version of Theorem 2.20. It will be proven in the following four sections.

Theorem 3.3 *Let $1/2 \leq a < 1$ and let $C_0 \in H_a(M; \Lambda^1 \otimes \mathfrak{k})$.*

i.) There exists $T > 0$ depending on C_0 (and not just on $\|C_0\|_{H_a}$. See Remark 3.16.) such that the integral equation (3.5) has a unique solution in \mathcal{P}_T^a .

ii.) If $1/2 < a < 1$ then the solution has finite strong a -action in the sense of (2.25).

iii.) If $a = 1/2$ and $\|C_0\|_{H_{1/2}}$ is sufficiently small then the solution has finite strong action in the sense of (2.25) with $a = 1/2$.

The proof of this theorem requires establishing properties of each of the terms on the right side of (3.5). Section 3.2 will show that the first term lies in \mathcal{P}_T^a . Section 3.3 will establish the needed contraction estimates for the second term. These will be put together in Section 3.4 to prove item *i.*, the existence and uniqueness portion of the theorem. Items *ii.* and *iii.*, finite action, will be proven in Section 3.5. Section (3.6) will show that solutions to the integral equation (3.5) are actually strong solutions.

3.2 Free propagation lies in the path space \mathcal{P}_T^a

We will show in this subsection that the first term on the right in (3.5) lies in \mathcal{P}_T^a and has finite strong a -action. All estimates in this subsection will be made for initial data $C_0 \in H_a$ with $0 \leq a < 1$ since there is no simplification for $a \geq 1/2$ and the greater generality will be needed later.

Lemma 3.4 *Let $0 \leq a < 1$ and suppose that $C_0 \in H_a$. Then, for some real constants c_a and γ_a there holds*

$$e^{2t}c_a\|C_0\|_{H_a}^2 \geq t^{1-a}\|e^{t\Delta}C_0\|_{H_1}^2 \rightarrow 0 \text{ as } t \downarrow 0 \quad \text{and} \quad (3.14)$$

$$\int_0^T t^{-a}\|e^{t\Delta}C_0\|_{H_1}^2 dt \leq e^{2T}\gamma_a^2\|C_0\|_{H_a}^2. \quad (3.15)$$

Proof. Denote by $E(d\lambda)$ the spectral resolution for the operator $1 - \Delta$ and let $\mu(d\lambda) = (E(d\lambda)D^aC_0, D^aC_0)$, where $D = \sqrt{1 - \Delta}$. In view of the definition (2.10) of the H_a norm we may write

$$\begin{aligned} e^{-2t}\|e^{t\Delta}C_0\|_{H_1}^2 &= \|De^{t(\Delta-1)}C_0\|_2^2 \\ &= \|D^{1-a}e^{-tD^2}D^aC_0\|_2^2 \\ &= (D^{2(1-a)}e^{-2tD^2}D^aC_0, D^aC_0) \\ &= \int_1^\infty \lambda^{(1-a)}e^{-2t\lambda}\mu(d\lambda). \end{aligned}$$

Hence

$$e^{-2t}t^{1-a}\|e^{t\Delta}C_0\|_{H_1}^2 = \int_1^\infty (t\lambda)^{1-a}e^{-2t\lambda}\mu(d\lambda). \quad (3.16)$$

The integrand is uniformly bounded in $\{t > 0 \text{ and } \lambda \geq 0\}$ by $c_a \equiv \sup_{\sigma>0} \sigma^{1-a}e^{-2\sigma}$ and therefore the integral is at most $c_a\|D^aC_0\|_2^2$. Moreover for each point $\lambda \in [0, \infty)$ the integrand goes to zero as $t \downarrow 0$. Since μ is a finite measure the dominated convergence theorem implies the remainder of (3.14).

Using now (3.16) again, and substituting $\tau = t\lambda$, we find

$$\begin{aligned}
e^{-2T} \int_0^T t^{-a} \|e^{t\Delta} C_0\|_{H_1}^2 dt &= e^{-2T} \int_0^T e^{2t} \int_1^\infty (t\lambda)^{-a} \lambda e^{-2t\lambda} \mu(d\lambda) dt \\
&\leq \int_1^\infty \int_0^T (t\lambda)^{-a} e^{-2t\lambda} \lambda dt \mu(d\lambda) \\
&= \int_1^\infty \left(\int_0^{T\lambda} \tau^{-a} e^{-2\tau} d\tau \right) \mu(d\lambda) \\
&\leq \gamma_a^2 \int_1^\infty \mu(d\lambda) = \gamma_a^2 \|D^a C_0\|_2^2,
\end{aligned}$$

where $\gamma_a^2 = \int_0^\infty \tau^{-a} e^{-2\tau} d\tau$. This proves (3.15). ■

Corollary 3.5 *Let $0 \leq a < 1$ and let $C_0 \in H_a$. Then the function*

$$[0, T] \ni t \mapsto C(t) := e^{t\Delta} C_0 \quad (3.17)$$

lies in \mathcal{P}_T^a for all $T \in (0, \infty)$.

Proof. $C(\cdot)$ is a continuous function on $[0, T]$ into H_a because $e^{t\Delta}$ is a strongly continuous semigroup in H_a . The second assertion in (3.14) shows that $t^{(1-a)/2} \|e^{t\Delta} C_0\|_{H_1} \rightarrow 0$ as $t \downarrow 0$, which is condition (3.7). Since $C(t) \in H_1$ for any $t > 0$, $C(\cdot)$ is also a continuous function on $(0, T]$ into H_1 . ■

Remark 3.6 (Pointwise behavior vs integral behavior) Let $f(t) = \|e^{t\Delta} C_0\|_{H_1}^2$. Observe that (3.14) says that $t^{-a} f(t) = o(t^{-1})$ while (3.15) says that $t^{-a} f(t)$ is integrable over $(0, T)$. Neither assertion implies the other. Both hold for this particular function. Both types of inequalities, pointwise in t and integral, will be needed for solutions $C(\cdot)$ to (2.22). Many of the apriori estimates that we will derive will show the strong interplay between them. This interplay was already a key tool in [2]. A pointwise inequality in t , such as (3.14) or (3.7), provides a mechanism for proving the existence of solutions for H_a initial data. But it is an integral condition, such as (3.15) or (2.25), or more particularly their gauge invariant version (2.14), which has direct physical significance and which we will address in a more gauge invariant formulation in a future work, [16].

3.3 Contraction estimates

For $C(\cdot)$ in the path space \mathcal{P}_T^a we have at our disposal two kinds of size conditions for use in estimating the terms in (3.3). $\|C(s)\|_{H_a}$ is continuous and therefore bounded on $[0, T]$, and therefore so also is $\|C(s)\|_{q_a}$, by Sobolev, where $q_a^{-1} = 1/2 - a/3$. (It may be useful to keep in mind that $q_{1/2} = 3$.) In addition, we have an s dependent bound on $\|C(s)\|_{H_1}$ of the form $\|C(s)\|_{H_1} \leq s^{(a-1)/2} \cdot |C|_T$, from (3.9). These two bounds will be used in different combinations. The following lemma lists several kinds of estimates that will be needed for the two different types of terms in $X(C)$. The proofs just rely on Hölder inequalities together with the Sobolev inequality $\|C\|_6 \leq \kappa_6 \|C\|_{H_1}$. We use $\|\partial C\|_2 \leq \|C\|_{H_1}$. We also continue to use, as in [2], the constant $c \equiv \sup\{\|ad\ x\|_{\mathfrak{k} \rightarrow \mathfrak{k}} : \|x\|_{\mathfrak{k}} \leq 1\}$, which measures the non-commutativity of \mathfrak{k} .

Lemma 3.7 *Let C be a \mathfrak{k} valued 1-form on M . Then the following inequalities hold. The Hölder inequality arithmetic needed in the proof is on the same line as the inequality. The power of c reflects the number of commutators that appear on the left.*

$$\|C^3\|_{6/5} \leq c^2 \kappa_6 \|C\|_{H_1} \|C\|_3^2 \quad 5/6 = 1/6 + 1/3 + 1/3 \quad (3.18)$$

$$\|C^3\|_{3/2} \leq c^2 \kappa_6^2 \|C\|_{H_1}^2 \|C\|_3 \quad 2/3 = 1/6 + 1/6 + 1/3 \quad (3.19)$$

$$\|C^3\|_2 \leq c^2 \kappa_6^3 \|C\|_{H_1}^3 \quad 1/2 = 1/6 + 1/6 + 1/6 \quad (3.20)$$

$$\|C \cdot \partial C\|_{6/5} \leq c \|C\|_{H_1} \|C\|_3 \quad 5/6 = 1/2 + 1/3 \quad (3.21)$$

$$\|C \cdot \partial C\|_{3/2} \leq c \kappa_6 \|C\|_{H_1}^2 \quad 2/3 = 1/6 + 1/2 \quad (3.22)$$

Proof. The proofs are in the right hand column. ■

Remark 3.8 The following elementary inequality is displayed here for frequent reference. If L is a non-negative self-adjoint operator on a Hilbert space and $D = L^{1/2}$ then

$$\|D^\alpha e^{-tL}\| \leq c_\alpha t^{-\alpha/2}, \quad t > 0, \quad \alpha \geq 0, \quad (3.23)$$

for some constant c_α , as follows from the spectral theorem and the inequality $\sup_{\lambda > 0} \lambda^{\alpha/2} e^{-t\lambda} = t^{-\alpha/2} \sup_{\sigma > 0} \sigma^{\alpha/2} e^{-\sigma}$. Here $\lambda \geq 0$ is a spectral parameter for L . The case of interest for us will be $L = 1 - \Delta$ acting on $L^2(M; \Lambda^1 \otimes \mathfrak{k})$.

Lemma 3.9 *Let $0 < a < 1$ and let $C \in \mathcal{P}_T^a$. Then*

$$\|C(s)^3\|_2 \leq s^{-(3/2)(1-a)} |C|_t^3 (c^2 \kappa_6^3) \quad \text{for } 0 < s \leq t \leq T \quad \text{and} \quad (3.24)$$

$$\|C(s) \cdot \partial C(s)\|_{3/2} \leq s^{-(1-a)} |C|_t^2 (c \kappa_6) \quad \text{for } 0 < s \leq t \leq T, \quad (3.25)$$

where $|C|_t$ is defined by (3.8). Further,

$$\|e^{(t-s)\Delta} \{C(s)^3\}\|_{H_1} \leq (t-s)^{-1/2} s^{3(a-1)/2} |C|_t^3 C_{40a}. \quad (3.26)$$

$$\|e^{(t-s)\Delta} \{C(s)^3\}\|_{H_a} \leq (t-s)^{-a/2} s^{3(a-1)/2} |C|_t^3 C_{41a}. \quad (3.27)$$

$$\|e^{(t-s)\Delta} \{C(s) \cdot \partial C(s)\}\|_{H_1} \leq (t-s)^{-3/4} s^{a-1} |C|_t^2 C_{42a}. \quad (3.28)$$

$$\|e^{(t-s)\Delta} \{C(s) \cdot \partial C(s)\}\|_{H_a} \leq (t-s)^{-(2a+1)/4} s^{a-1} |C|_t^2 C_{43a}. \quad (3.29)$$

The constants C_{ja} depend only on Sobolev constants, on the constants c_α in (3.23), on powers of the commutator norm c in \mathfrak{k} and on a .

Proof. By (3.20) and (3.9) we have

$$\|C(s)^3\|_2 \leq c^2 \kappa_6^3 \|C(s)\|_{H_1}^3 \leq c^2 \kappa_6^3 (s^{(a-1)/2} |C|_t)^3,$$

which is (3.24). Combining (3.22) and (3.9), one finds

$$\|C(s) \cdot \partial C(s)\|_{3/2} \leq c \kappa_6 \|C(s)\|_{H_1}^2 \leq c \kappa_6 (s^{(a-1)/2} |C|_t)^2,$$

which is (3.25).

The two inequalities (3.26) and (3.27) follow directly from (3.24) combined with (3.23) with $\alpha = 1$ or a , respectively. Here we are ignoring irrelevant factors of e^T needed to justify $\|e^{r\Delta} f\|_{H_\alpha} = \|D^\alpha e^{r\Delta} f\|_2$ because we are only concerned with small T .

For the remaining two inequalities we need to interpose a Sobolev inequality before applying (3.23). We have a bound, κ' say, on the norm of $D^{-1/2} : L^{3/2} \rightarrow L^2$ because $1/2 = (2/3) - (1/2)/3$. Thus we can write

$$\begin{aligned} \|e^{(t-s)\Delta} \{C(s) \cdot \partial C(s)\}\|_{H_1} &= \|D^{1/2} e^{(t-s)\Delta} D^{-1/2} \{C(s) \cdot \partial C(s)\}\|_{H_1} \\ &= \|D^{3/2} e^{(t-s)\Delta} D^{-1/2} \{C(s) \cdot \partial C(s)\}\|_2 \\ &\leq \|D^{3/2} e^{(t-s)\Delta}\|_{2 \rightarrow 2} \|D^{-1/2} \{C(s) \cdot \partial C(s)\}\|_2 \\ &\leq \left(c_{3/2} (t-s)^{-3/4} \right) \cdot \left(\kappa' s^{a-1} |C|_t^2 (c \kappa_6) \right), \end{aligned}$$

wherein we have used (3.25) and (3.23). This proves (3.28). The proof of (3.29) is the same but with H_1 replaced by H_a and with $D^{3/2}$ replaced by $D^{1/2+a}$ in the second and third lines. In this case α in (3.23) should be taken to be $1/2 + a$, giving (3.29) ■

Remark 3.10 The following identity, which arises frequently, is listed here for convenience. Let μ and ν be real numbers with $\mu < 1$ and $\nu < 1$. Then

$$\frac{1}{t} \int_0^t (t-s)^{-\mu} s^{-\nu} ds = t^{-\mu-\nu} C_{\mu,\nu} \quad (3.30)$$

for some finite constant $C_{\mu,\nu}$. For the proof, make the change of variables $s = tr$ to convert the integral to $t^{-\mu-\nu} \int_0^1 (1-r)^{-\mu} r^{-\nu} dr$, which has the asserted form.

Lemma 3.11 *Let $1/2 \leq a < 1$ and let $C(\cdot)$ be in \mathcal{P}_T^a . Define*

$$w(t) = \int_0^t e^{(t-s)\Delta} X(C(s)) ds. \quad (3.31)$$

Then

$$w : [0, T] \rightarrow H_a \quad \text{and} \quad w : (0, T] \rightarrow H_1 \quad (3.32)$$

are both continuous. Moreover

$$t^{\frac{1-a}{2}} \|w(t)\|_{H_1} \leq \left(t^{a-(1/2)} |C|_t^3 + t^{\frac{a-(1/2)}{2}} |C|_t^2 \right) C_{50a}, \quad (3.33)$$

$$\|w(t)\|_{H_a} \leq \left(t^{a-(1/2)} |C|_t^3 + t^{\frac{a-(1/2)}{2}} |C|_t^2 \right) C_{51a} \quad \text{and} \quad (3.34)$$

$$\|w(t)\|_{q_a} \leq \left(t^{a-(1/2)} |C|_t^3 + t^{\frac{a-(1/2)}{2}} |C|_t^2 \right) C_{52a}, \quad (3.35)$$

where $q_a^{-1} = (1/2) - (a/3)$.

The constants C_{ja} depend only on Sobolev constants, the coefficients c_α in (3.23), on the commutator norm c and on a .

Proof. The sums on the right sides of these three inequalities correspond to the decomposition $X(C) = C^3 + C \cdot \partial C$ in (3.3). We need to carry out the derivation of these inequalities separately for the cases $C(s)^3$ and $C(s) \cdot \partial C(s)$ because of the slightly different powers of $(t-s)$ and s that occur in (3.26) - (3.29). The following derivation is typical of all of them. We have

$$\int_0^t \|e^{(t-s)\Delta} C(s)^3\|_{H_1} ds \leq \int_0^t (t-s)^{-1/2} s^{3(a-1)/2} ds |C|_t^3 C_{40a} \quad (3.36)$$

by (3.26). All the other three estimates needed in (3.33) and (3.34) have a similar form. They differ only in the powers $(t-s)^{-\mu} s^{-\nu}$ that occur. The

identity (3.30) shows that $\int_0^t (t-s)^{-\mu} s^{-\nu} ds = t^{1-\mu-\nu} \cdot \text{constant}$. Thus in the case of (3.36) one sees that $1-\mu-\nu = 1-(1/2)+3(a-1)/2 = -1+(3/2)a$. This gives correctly the power for the first term on the right in (3.33) upon taking into account the factor $t^{(1-a)/2}$ on the left side of (3.33). We leave the arithmetic for the remaining three cases to the reader. By Sobolev, (3.35) follows from (3.34).

It remains to prove the two assertions about continuity in (3.32). Observe first that (3.34) implies continuity of w into H_a at $t = 0$ because $w(0) = 0$ and $|C|_t \rightarrow 0$ as $t \downarrow 0$ by (3.11). (Notice that for $a = 1/2$ we must rely on $|C|_t \rightarrow 0$ whereas for $a > 1/2$ the strictly positive powers of t in (3.34) are enough to ensure that $\|w(t)\|_{H_a} \rightarrow 0$.) It suffices, therefore, to prove both continuities on an interval $[\epsilon, T]$ with $\epsilon > 0$. Suppose then that $0 < \epsilon \leq r < t \leq T$. The identity

$$w(t) - w(r) = \int_r^t e^{(t-s)\Delta} F(s) ds + \int_0^r \left(e^{(t-r)\Delta} - I \right) e^{(r-s)\Delta} F(s) ds, \quad (3.37)$$

wherein $F(s) = X(C(s))$ is easily verified. We need to show that $\|w(t) - w(r)\|_{H_\alpha} \rightarrow 0$ as $t - r \rightarrow 0$ in the interval $[\epsilon, T]$ for $\alpha = 1$ and $\alpha = a$. First consider the term $F(s) = C(s)^3$ in $X(C(s))$. We have, by (3.26) and (3.27),

$$\int_r^t \|e^{(t-s)\Delta} C(s)^3\|_{H_\alpha} ds \leq \begin{cases} \int_r^t (t-s)^{-1/2} s^{3(a-1)/2} ds |C|_T^3 C_{40}, & \alpha = 1 \\ \int_r^t (t-s)^{-a/2} s^{3(a-1)/2} ds |C|_T^3 C_{41}, & \alpha = a. \end{cases}$$

Both integrals on the right go to zero as $t - r \rightarrow 0$ if r and t are bounded away from zero. Similarly, by (3.28) and (3.29),

$$\int_r^t \|e^{(t-s)\Delta} \{C(s) \cdot \partial C(s)\}\|_{H_\alpha} ds \leq \begin{cases} \int_r^t (t-s)^{-3/4} s^{a-1} ds |C|_T^2 C_{42}, & \alpha = 1 \\ \int_r^t (t-s)^{-(2a+1)/4} s^{a-1} ds |C|_T^2 C_{43}, & \alpha = a, \end{cases}$$

which also goes to zero if r and t lie in the interval $[\epsilon, T]$ and $t - r \rightarrow 0$.

Concerning the second integral in (3.37) observe that, although the operator in parentheses goes to zero strongly as $t - r \downarrow 0$, it does not go to zero in norm. Let $0 < \delta < 1/4$. Then, for any measurable function

$F : (0, T] \rightarrow L^2(M; \Lambda^1 \otimes \mathfrak{k})$, we have

$$\begin{aligned}
& \int_0^r \| (e^{(t-r)\Delta} - I) e^{(r-s)\Delta} F(s) \|_{H_\alpha} ds \\
&= \int_0^r \| (e^{(t-r)\Delta} - I) D^{-2\delta} D^{2\delta} e^{(r-s)\Delta} F(s) \|_{H_\alpha} ds \\
&\leq \| (e^{(t-r)\Delta} - I) D^{-2\delta} \|_{2 \rightarrow 2} \int_0^r \| D^{2\delta} e^{(r-s)\Delta} F(s) \|_{H_\alpha} ds. \quad (3.38)
\end{aligned}$$

The operator norm in the first factor goes to zero for any $\delta > 0$ as $t - r \downarrow 0$. It suffices to prove therefore that the integral factor is uniformly bounded for $r \in [\epsilon, T]$. But (3.23) implies that, in the presence of the factor $D^{2\delta}$, each of the factors $(t - s)^{-\mu}$ in the inequalities (3.26) - (3.29) need only be replaced by $(t - s)^{-\mu-\delta}$. All of these four exponents remain greater than -1 for $1/2 \leq a \leq 1$ because $\delta < 1/4$. Consequently the four estimates needed to bound the integral factor in (3.38), for $\alpha = 1$ or a and $F = C^3$ or $C \cdot \partial C$, remain bounded on the interval $\epsilon \leq r \leq T$. ■

3.4 Proof of existence of mild solutions

Notation 3.12 Let $1/2 \leq a < 1$ and suppose that $C_0 \in H_a(M)$. Let

$$\mathcal{P}_{T,b}^a = \{C(\cdot) \in \mathcal{P}_T^a : |C|_T \leq b\}, \quad (3.39)$$

where $|C|_t$ is defined as in (3.8). For any $b > 0$ the set $\mathcal{P}_{T,b}^a$ is complete in the metric (3.12) and is non-empty for some $T > 0$ since, by Corollary 3.5, $\mathcal{P}_{T_1}^a$ is non-empty for all $T_1 > 0$, and if $C(\cdot) \in \mathcal{P}_{T_1}^a$ then the restriction of $C(\cdot)$ to $[0, T]$ will be in $\mathcal{P}_{T,b}^a$ for some $T \in (0, T_1]$ by virtue of (3.11). It is this feature of the spaces \mathcal{P}_T^a that will allow our method to work in the critical case $a = 1/2$. See Remark 3.17 for further discussion of this.

Lemma 3.13 *Define*

$$Z(C)(t) = e^{t\Delta} C_0 + \int_0^t e^{(t-s)\Delta} X(C(s)) ds \quad \text{for } C(\cdot) \in \mathcal{P}_T^a. \quad (3.40)$$

Then

$$Z(\mathcal{P}_T^a) \subset \mathcal{P}_T^a. \quad (3.41)$$

Let

$$b_0 = \sup_{0 < t \leq T} t^{(1-a)/2} \|e^{t\Delta} C_0\|_{H_1}.$$

If $C \in \mathcal{P}_{T,b}^a$ and $T \leq 1$ then

$$|Z(C)|_T \leq b_0 + (b^2 + b^3)C_{50a}, \quad (3.42)$$

where C_{50a} is defined in (3.33). If $b > 0$ is chosen so small that

$$(b^2 + b^3)C_{50a} \leq b/2 \quad (3.43)$$

and $T > 0$ is chosen so small that

$$b_0 \leq b/2 \quad (3.44)$$

then Z takes $\mathcal{P}_{T,b}^a$ into itself.

Proof. By Corollary 3.5 the first term in (3.40) lies in \mathcal{P}_T^a . By Lemma 3.11 the second term in (3.40) defines a continuous function on $[0, T]$ into H_a and a continuous function on $(0, T]$ into H_1 . Further, (3.33) shows that condition (3.7) holds for the second term because the factors $|C|_t$ go to zero as $t \downarrow 0$ in accordance with (3.11). This proves (3.41).

It is worth noting a distinction between $a = 1/2$ and $a > 1/2$ that will recur often: For $a = 1/2$ the right side of (3.33) goes to zero as $t \downarrow 0$ only because $|C|_t \rightarrow 0$, which is built into the definition of \mathcal{P}_T^a . For $a > 1/2$ the two strictly positive powers of t on the right side of (3.33) contribute further to the decay as $t \downarrow 0$.

In view of the definition (3.8), the inequality (3.42) follows from (3.33) (with $0 \leq t \leq T \leq 1$) and the definition of b_0 . Now if b is chosen so small that (3.43) holds then (3.42) shows that $|Z(C)|_T \leq b_0 + (b/2)$. Further, (3.14) shows that we can choose $T > 0$ so small that (3.44) holds. Thus for these values of b and T we find $|Z(C)|_T \leq b$. Therefore Z takes $\mathcal{P}_{T,b}^a$ into itself. ■

Lemma 3.14 Z is a contraction on $\mathcal{P}_{T,b}^a$ for b and T sufficiently small.

Proof. If $C_j \in \mathcal{P}_{T,b}^a$ for $j = 1, 2$ then, for $0 < t \leq T \leq 1$,

$$\begin{aligned} & t^{(1-a)/2} \|Z(C_1)(t) - Z(C_2)(t)\|_{H_1} \\ & \leq t^{(1-a)/2} \int_0^t \|e^{(t-s)\Delta} \{X(C_1(s)) - X(C_2(s))\}\|_{H_1} ds \\ & \leq |C_1 - C_2|_T (2b + 3b^2) C_{50a} \end{aligned} \quad (3.45)$$

by polarization of (3.33) (with $t = T$ in that inequality.) Similarly, by polarizing (3.34) we find, for $0 \leq t \leq T \leq 1$,

$$\|Z(C_1)(t) - Z(C_2)(t)\|_{H_a} \leq |C_1 - C_2|_T(2b + 3b^2)C_{51a}. \quad (3.46)$$

Choose b so small that not only (3.43) holds, but also

$$(2b + 3b^2) \max(C_{50a}, C_{51a}) \leq 1/4. \quad (3.47)$$

Since C_{50a} and C_{51a} are independent of C_0 and T , so is the size restriction on b . As we saw in Lemma 3.13, for our fixed $C_0 \in H_a$, we can choose T so small that (3.44) holds. For such choices of b and T , Z takes $\mathcal{P}_{T,b}^a$ into itself by Lemma 3.13 and

$$\begin{aligned} \text{dist}(Z(C_1), Z(C_2)) &= \sup_{0 \leq t \leq T} \|Z(C_1)(t) - Z(C_2)(t)\|_{H_a} \\ &\quad + \sup_{0 \leq t \leq T} t^{(1-a)/2} \|Z(C_1)(t) - Z(C_2)(t)\|_{H_1} \\ &\leq (1/4)|C_1 - C_2|_T + (1/4)|C_1 - C_2|_T \\ &\leq (1/2)\text{dist}(C_1, C_2) \end{aligned} \quad (3.48)$$

by (3.45) and (3.46). Thus Z is a contraction on $\mathcal{P}_{T,b}^a$. ■

Remark 3.15 The fact that $\sup_{0 < t \leq T} \|C_1(t) - C_2(t)\|_{H_a}$ does not enter into the right sides of (3.45) or (3.46) suggests that, in some sense, the behavior (3.7), of $\|C(s)\|_{H_1}$ near $s = 0$, controls $\|C(t)\|_{H_a}$. We will see strong forms of this in the papers [15] and [16].

Proof of Theorem 3.3, Part i.). The proof is an immediate consequence of Lemma 3.14. ■

Remark 3.16 (Dependence of T on C_0) The time T that we have produced in Theorem 3.3 depends on C_0 itself, and not just on $\|C_0\|_{H_a}$, because the strong limit in (3.14) cannot be replaced by a limit in operator norm. Indeed, (3.14) asserts that the operator function $t \mapsto (t^{(1-a)/2}e^{t\Delta} : H_a \rightarrow H_1)$ goes to zero strongly as $t \downarrow 0$. One can verify with the help of the spectral theorem that it does not go to zero in operator norm.

Remark 3.17 Typically, a proof of existence and uniqueness of solutions for the integral equation (3.5) proceeds by establishing estimates for the non-linear operator Z in (3.40) of the form

$$\|Z(C_1(\cdot)) - Z(C_2(\cdot))\| \leq \text{const.} T^\alpha \|C_1(\cdot) - C_2(\cdot)\|, \quad (3.49)$$

for some $\alpha > 0$ and some norm on a Banach space containing the freely propagated term $e^{t\Delta}C_0$ in (3.5). One need only take T small to conclude that Z is a contraction. Typical estimation methods for establishing (3.49) in some contexts can be found, for example, in Taylor, [38, page 273]. In our context such a contraction proof works in case $a > 1/2$. Indeed polarization of (3.33) and (3.34) shows that for $0 < t \leq T \leq 1$ one can include a factor $T^{\frac{a-(1/2)}{2}}$ on the right hand sides of (3.45) and (3.46). Thus (3.49) holds with $\alpha = (a - (1/2))/2$. Since $\alpha > 0$ when $a > 1/2$ one could proceed in this case in the usual way without having to rely on the fact that $t^{(1-a)/2}\|C(t)\|_{H_1}$ is not only bounded, but also goes to zero as $t \downarrow 0$, as was assumed in (3.7). In case $a = 1/2$ one has $\alpha = 0$ and the preceding standard technique for proving contraction fails. The requirement (3.7) then becomes essential in the choice of the path space \mathcal{P}_T^a . It is not clear whether this distinction in techniques for $a > 1/2$ or $a = 1/2$ is an intrinsic feature of criticality or an artifact of our choice of metric space \mathcal{P}_T^a . We will see a similar dichotomy in dealing with finite action in the next subsection.

3.5 $C(\cdot)$ has finite action

Theorem 3.18 *Let $1/2 \leq a < 1$. Suppose that $C_0 \in H_a$. Let $C(\cdot)$ be the solution to the integral equation (3.5) produced in Theorem 3.3, Part i.). If $a > 1/2$ then*

$$\int_0^T s^{-a} \|C(s)\|_{H_1}^2 ds < \infty \quad \text{for sufficiently small } T. \quad (3.50)$$

If $a = 1/2$ and $\|C_0\|_{H_{1/2}}$ is sufficiently small then

$$\int_0^T s^{-1/2} \|C(s)\|_{H_1}^2 ds < \infty \quad \text{for sufficiently small } T. \quad (3.51)$$

The proof will be developed in the next two subsections.

3.5.1 Abstract action estimates

We need to make an estimate of the integral term in (3.5) similar to the estimate (3.15) for the freely propagated term. We will be able to avoid using heat kernel estimates in favor of just the spectral theorem and simple Sobolev inequalities with the help of the following theorem. The parameters α, μ, b in the theorem will be chosen to fit our various needs in this paper and its sequel. The operator L of interest to us will be $1 - \Delta$ on forms.

Theorem 3.19 *Let L be a non-negative self-adjoint operator on a Hilbert space \mathcal{H} . Suppose that α, μ, b are real numbers such that*

$$0 \leq \alpha \leq 1, \quad (3.52)$$

$$0 \leq \mu \leq b < 1, \quad (3.53)$$

$$\delta \equiv 1 - \alpha - \mu \geq 0. \quad (3.54)$$

Then there is a constant $C_{\alpha, \mu}$, depending only on α and μ , such that $C_{\alpha, 0} \leq 1$ and such that for $0 < T < \infty$ and for any measurable function $g : (0, T) \rightarrow \mathcal{H}$ there holds

$$\int_0^T t^{-b} \left\| \int_0^t s^{-\mu} L^\alpha e^{-(t-s)L} g(s) ds \right\|^2 dt \leq T^{2\delta} \int_0^T s^{-b} \|g(s)\|^2 ds \cdot C_{\alpha, \mu}. \quad (3.55)$$

In particular, for $\mu = 0$, there holds

$$\int_0^T t^{-b} \left\| \int_0^t L^\alpha e^{-(t-s)L} g(s) ds \right\|^2 dt \leq T^{2(1-\alpha)} \int_0^T s^{-b} \|g(s)\|^2 ds \quad \text{for } 0 \leq b < 1. \quad (3.56)$$

The proof depends on the following lemmas.

Lemma 3.20 (*Schwarz-like inequality*) *Suppose that $(0, t) \ni s \mapsto H(s)$ is strongly continuous function into a set of commuting, bounded, non-negative Hermitian operators on a Hilbert space \mathcal{H} . If $g : (0, T) \rightarrow \mathcal{H}$ is measurable then*

$$\left\| \int_0^t H(s) g(s) ds \right\|^2 \leq \left\| \int_0^t H(s) ds \right\| \int_0^t (H(s) g(s), g(s)) ds \quad (3.57)$$

Proof. Let $F(s) = H(s)^{1/2}$. Then

$$\begin{aligned}
\left\| \int_0^t H(s)g(s)ds \right\|^2 &= \left\| \int_0^t F(s)^2 g(s)ds \right\|^2 \\
&= \int_0^t ds_1 \int_0^t ds_2 \left(F(s_1)^2 g(s_1), F(s_2)^2 g(s_2) \right) \\
&= \int_0^t ds_1 \int_0^t ds_2 \left(F(s_2)F(s_1)g(s_1), F(s_1)F(s_2)g(s_2) \right) \\
&\leq (1/2) \int_0^t ds_1 \int_0^t ds_2 \left(\|F(s_2)F(s_1)g(s_1)\|^2 + \|F(s_1)F(s_2)g(s_2)\|^2 \right) \\
&= \int_0^t ds_1 \int_0^t ds_2 \|F(s_2)F(s_1)g(s_1)\|^2 \\
&= \int_0^t ds_1 \int_0^t ds_2 \left(F(s_2)^2 F(s_1)g(s_1), F(s_1)g(s_1) \right) \\
&= \int_0^t ds_1 \left(\left\{ \int_0^t ds_2 F(s_2)^2 \right\} F(s_1)g(s_1), F(s_1)g(s_1) \right) \\
&\leq \left\| \int_0^t F(s_2)^2 ds_2 \right\| \int_0^t ds_1 \left(F(s_1)g(s_1), F(s_1)g(s_1) \right).
\end{aligned}$$

■

Remark 3.21 A simpler and cruder proof would easily give the inequality (3.57) but with $\|H(s)\|$ under the integral in the first factor. However in the case of interest to us this integral would diverge.

Lemma 3.22 *Under the hypotheses of Theorem 3.19 we have*

$$\left\| \int_0^t s^{-\mu} L^\alpha e^{-(t-s)L} ds \right\| \leq t^\delta (1 - \mu)^{\alpha-1}. \quad (3.58)$$

Proof. By the spectral theorem we need only prove that for any $\lambda > 0$ there holds

$$\int_0^t s^{-\mu} \lambda^\alpha e^{-(t-s)\lambda} ds \leq t^{1-\alpha-\mu} (1 - \mu)^{\alpha-1}. \quad (3.59)$$

Make the substitution $r = s\lambda$ and define $\gamma = t\lambda$ to find

$$\begin{aligned}
\int_0^t s^{-\mu} \lambda^\alpha e^{-(t-s)\lambda} ds &= e^{-\gamma} \int_0^\gamma r^{-\mu} \lambda^{\alpha+\mu-1} e^r dr \\
&= t^{1-\alpha-\mu} \left\{ \gamma^{\alpha+\mu-1} e^{-\gamma} \int_0^\gamma r^{-\mu} e^r dr \right\}.
\end{aligned}$$

It suffices to prove that the expression in braces is at most $(1 - \mu)^{\alpha-1}$ for all $\gamma > 0$. In case $\gamma \leq 1 - \mu$ the expression in braces is at most

$$\gamma^{\alpha+\mu-1} \int_0^\gamma r^{-\mu} dr = \gamma^{\alpha+\mu-1} \frac{\gamma^{1-\mu}}{1-\mu} = \frac{\gamma^\alpha}{1-\mu} \leq \frac{(1-\mu)^\alpha}{1-\mu}.$$

This proves (3.59) when $t\lambda \leq 1 - \mu$. In case $\gamma > 1 - \mu$ the expression in braces is at most, considering that $\alpha + \mu - 1 \leq 0$,

$$\begin{aligned} & (1 - \mu)^{\alpha+\mu-1} e^{-\gamma} \int_0^\gamma r^{-\mu} e^r dr \\ & \leq (1 - \mu)^{\alpha+\mu-1} e^{-\gamma} \left(\int_0^{1-\mu} r^{-\mu} dr e^{1-\mu} + (1 - \mu)^{-\mu} \int_{1-\mu}^\gamma e^r dr \right) \\ & = (1 - \mu)^{\alpha+\mu-1} e^{-\gamma} \left((1 - \mu)^{-\mu} e^{1-\mu} + (1 - \mu)^{-\mu} (e^\gamma - e^{1-\mu}) \right) \\ & = (1 - \mu)^{\alpha+\mu-1} e^{-\gamma} (1 - \mu)^{-\mu} e^\gamma. \end{aligned}$$

This proves (3.59) when $t\lambda > 1 - \mu$. ■

Lemma 3.23 *Under the hypotheses of Theorem 3.19 we have*

$$\begin{aligned} & \left\| \int_0^t s^{-\mu} L^\alpha e^{-(t-s)L} g(s) ds \right\|^2 \\ & \leq (1 - \mu)^{\alpha-1} t^\delta \int_0^t \left(s^{-\mu} L^\alpha e^{-(t-s)L} g(s), g(s) \right) ds. \end{aligned} \quad (3.60)$$

Proof. Let $H(s) = s^{-\mu} L^\alpha e^{-(t-s)L}$, $0 < s < t$. Then (3.58) shows that $\| \int_0^t H(s) ds \| \leq t^\delta (1 - \mu)^{\alpha-1}$. Insert this bound into (3.57) to find (3.60). ■

We will need also an estimate of the following integral over (s, T) .

Lemma 3.24 *Under the hypotheses of Theorem 3.19 we have*

$$\left\| \int_s^T t^{-b} t^\delta L^\alpha e^{-(t-s)L} dt \right\| \leq s^{\mu-b} T^{2\delta} C(\mu, \delta) \quad (3.61)$$

for some finite constant $C(\mu, \delta)$ with $C(0, \delta) \leq 1$. In particular, for $\mu = 0$, there holds

$$\left\| \int_s^T t^{-b} t^\delta L^\alpha e^{-(t-s)L} dt \right\| \leq s^{-b} T^{2\delta} \quad \text{for } 0 \leq b < 1, \quad (3.62)$$

$$\left\| \int_s^T t^{-b} L^\alpha e^{-(t-s)L} dt \right\| \leq s^{-b} T^\delta \quad \text{for } 0 \leq b < 1, \quad (3.63)$$

where $\delta = 1 - \alpha$.

Proof. By the spectral theorem it suffices to show that

$$\int_s^T t^{-b} t^\delta \lambda^\alpha e^{-(t-s)\lambda} dt \leq s^{\mu-b} T^{2\delta} C(\mu, \delta) \quad (3.64)$$

for some finite function $C(\cdot, \cdot)$ with $C(0, \delta) \leq 1$. Observe that $t^{-b} t^\delta = t^{\mu-b} t^\delta t^{-\mu} \leq s^{\mu-b} T^\delta t^{-\mu}$ for $s \leq t \leq T$ because $\mu - b \leq 0$ and $\delta \geq 0$. Hence

$$\int_s^T t^{-b} t^\delta \lambda^\alpha e^{-(t-s)\lambda} dt \leq s^{\mu-b} T^\delta \int_s^T t^{-\mu} \lambda^\alpha e^{-(t-s)\lambda} dt. \quad (3.65)$$

To estimate the last integral make the change of variables $t = s + (\sigma/\lambda)$ in the integral to find

$$\begin{aligned} \int_s^T t^{-\mu} \lambda^\alpha e^{-(t-s)\lambda} dt &= \int_0^{(T-s)\lambda} \left(s + \frac{\sigma}{\lambda}\right)^{-\mu} \lambda^{\alpha-1} e^{-\sigma} d\sigma \\ &\leq \int_0^{T\lambda} \left(\frac{\sigma}{\lambda}\right)^{-\mu} \lambda^{\alpha-1} e^{-\sigma} d\sigma \\ &= \int_0^{T\lambda} \sigma^{-\mu} \lambda^{-\delta} e^{-\sigma} d\sigma \\ &= T^\delta (T\lambda)^{-\delta} \int_0^{T\lambda} \sigma^{-\mu} e^{-\sigma} d\sigma \\ &\leq T^\delta \sup_{\tau>0} \tau^{-\delta} \int_0^\tau \sigma^{-\mu} e^{-\sigma} d\sigma. \end{aligned}$$

It remains, therefore, only to show that the function

$$C(\mu, \delta) \equiv \sup_{\tau>0} \tau^{-\delta} \int_0^\tau \sigma^{-\mu} e^{-\sigma} d\sigma \quad (3.66)$$

is finite for the allowed values of μ and δ and is at most one at $\mu = 0$. Since $\delta \geq 0$ and $\mu < 1$ we have

$$\limsup_{\tau \rightarrow \infty} \left(\tau^{-\delta} \int_0^\tau \sigma^{-\mu} e^{-\sigma} d\sigma \right) \leq \left(\limsup_{\tau \rightarrow \infty} \tau^{-\delta} \right) \int_0^\infty \sigma^{-\mu} e^{-\sigma} d\sigma < \infty.$$

For small τ we have

$$\tau^{-\delta} \int_0^\tau \sigma^{-\mu} e^{-\sigma} d\sigma \leq \tau^{-\delta} \int_0^\tau \sigma^{-\mu} d\sigma = \tau^{-\delta} \tau^{1-\mu} / (1-\mu) = \tau^{1-\delta-\mu} / (1-\mu),$$

which is bounded for small τ because $1 - \delta - \mu = \alpha \geq 0$. Thus $C(\mu, \delta) < \infty$. Finally, if $\mu = 0$ then $\delta = 1 - \alpha$ and $C(0, \delta) = \sup_{\tau > 0} \left\{ \tau^\alpha \tau^{-1} \int_0^\tau e^{-\sigma} d\sigma \right\}$. For $\tau \geq 1$ the expression in braces is increasing in α and for $\alpha = 1$ is at most one, while for $\tau < 1$ it is decreasing in α and for $\alpha = 0$ is at most one. Therefore the expression in braces is at most one for all $\tau > 0$. ■

Proof of Theorem 3.19. By (3.60) and (3.61) we have

$$\begin{aligned}
& \int_0^T t^{-b} \left\| \int_0^t s^{-\mu} L^\alpha e^{-(t-s)L} g(s) ds \right\|^2 dt \\
& \leq (1 - \mu)^{\alpha-1} \int_0^T t^{-b} t^\delta \int_0^t \left(s^{-\mu} L^\alpha e^{-(t-s)L} g(s), g(s) \right) ds dt \\
& = (1 - \mu)^{\alpha-1} \int_0^T s^{-\mu} \int_s^T t^{\delta-b} \left(L^\alpha e^{-(t-s)L} g(s), g(s) \right) dt ds \\
& = (1 - \mu)^{\alpha-1} \int_0^T s^{-\mu} \left(\left\{ \int_s^T t^{\delta-b} L^\alpha e^{-(t-s)L} dt \right\} g(s), g(s) \right) ds \\
& \leq (1 - \mu)^{\alpha-1} \int_0^T s^{-\mu} \left\| \int_s^T t^{\delta-b} L^\alpha e^{-(t-s)L} dt \right\| \|g(s)\|^2 ds \\
& \leq (1 - \mu)^{\alpha-1} \int_0^T s^{-\mu} \{ s^{\mu-b} T^{2\delta} C(\mu, \delta) \} \|g(s)\|^2 ds.
\end{aligned}$$

Thus we may take $C_{\alpha, \mu} = (1 - \mu)^{\alpha-1} C(\mu, \delta)$ to arrive at (3.55). Since $C(\mu, \delta) \leq 1$ if $\mu = 0$, (3.56) holds. ■

3.5.2 Proof of finite action

Action estimates for the freely propagated term in (3.5) have been made in Lemma 3.4 for all $a \in (0, 1)$. In this section action estimates will be made for the integral term in (3.5) for $1/2 \leq a < 1$.

Lemma 3.25 *Define*

$$D = (1 - \Delta)^{1/2}.$$

Let $6/5 \leq p \leq 2$. Define $\gamma \in [0, 1]$ by the condition

$$1/2 = p^{-1} - \gamma/3. \tag{3.67}$$

Let κ_p denote the norm of $D^{-\gamma}$ as an operator from $L^p(M; \Lambda^1 \otimes \mathfrak{k})$ into $L^2(M; \Lambda^1 \otimes \mathfrak{k})$. By Sobolev this is finite. Let $0 < b < 1$ and let $f : (0, T) \rightarrow L^p(M; \Lambda^1 \otimes \mathfrak{k})$ be a measurable function. Then

$$\int_0^T t^{-b} \left\| \int_0^t e^{(t-s)\Delta} f(s) ds \right\|_{H_1}^2 dt \leq T^{1-\gamma} \int_0^T s^{-b} \|f(s)\|_p^2 ds \cdot (e^{2T} \kappa_p^2). \quad (3.68)$$

Proof. Let $g(s) = e^{-s} D^{-\gamma} f(s)$. Then $\|g(s)\|_2 \leq \kappa_p \|f(s)\|_p$. We are going to apply Theorem 3.19 with $L = (1 - \Delta) = D^2$, $\mu = 0$ and $2\alpha = 1 + \gamma$. Then $2\delta = 2 - 2\alpha = 1 - \gamma$. Since $f(s) = e^s D^\gamma g(s)$ we have, using (3.56) in the fourth line,

$$\begin{aligned} \int_0^T t^{-b} \left\| \int_0^t e^{(t-s)\Delta} f(s) ds \right\|_{H_1}^2 dt &= \int_0^T t^{-b} \left\| \int_0^t D e^{-(t-s)(1-\Delta)} e^{(t-s)} f(s) ds \right\|_2^2 dt \\ &= \int_0^T t^{-b} \left\| e^t \int_0^t D^{1+\gamma} e^{-(t-s)(1-\Delta)} g(s) ds \right\|_2^2 dt \\ &= \int_0^T t^{-b} e^{2t} \left\| \int_0^t L^\alpha e^{-(t-s)L} g(s) ds \right\|_2^2 dt \\ &\leq T^{2\delta} \int_0^T s^{-b} \|g(s)\|_2^2 ds e^{2T} \\ &\leq T^{2\delta} \int_0^T s^{-b} \|f(s)\|_p^2 ds (e^{2T} \kappa_p^2). \end{aligned}$$

■

Lemma 3.26 Let $1/2 \leq a < 1$. Define

$$q_a^{-1} = (1/2) - (a/3), \quad p_a^{-1} = (7/6) - (2a/3), \quad r_a^{-1} = 1 - (a/3). \quad (3.69)$$

Let $C(\cdot) \in \mathcal{P}_T^a$ and define

$$\beta_a = \sup_{0 \leq s \leq T} \|C(s)\|_{q_a}. \quad (3.70)$$

Then, for $0 < s \leq T$,

$$\|C(s)^3\|_{p_a} \leq c^2 \kappa_6 \beta_a^2 \|C(s)\|_{H_1}, \quad p_a^{-1} = 2q_a^{-1} + (1/6) \quad (3.71)$$

$$\|C(s) \cdot \partial C(s)\|_{r_a} \leq c \beta_a \|C(s)\|_{H_1}, \quad r_a^{-1} = q_a^{-1} + (1/2). \quad (3.72)$$

Proof. Since $\|C(s)\|_6 \leq \kappa_6 \|C(s)\|_{H_1}$ and $\|\partial C(s)\|_2 \leq \|C(s)\|_{H_1}$, Hölder's inequality proves (3.71) and (3.72) in accordance with the arithmetic shown in the second column. ■

Remark 3.27 It may be clarifying to contrast the critical case $a = 1/2$ with the case $a > 1/2$. The definitions (3.69) give

$$\begin{aligned} q_a &= 3, & p_a &= r_a = 6/5 & \text{for } a &= 1/2 \\ 3 < q_a < 6, & 6/5 < p_a < 2, & 6/5 < r_a < 3/2 & \text{for } 1/2 < a < 1. \end{aligned}$$

For the two types of terms that appear on the left side in (3.71) and (3.72) we are going to apply Lemma 3.25 with $p = p_a$ and $p = r_a$, respectively. With γ defined by (3.67) and with the help of some arithmetic, the reader can verify that the corresponding values of γ yield

$$1 - \gamma = \begin{cases} 2(a - \frac{1}{2}) & \text{if } p = p_a \\ (a - \frac{1}{2}) & \text{if } p = r_a. \end{cases} \quad (3.73)$$

Lemma 3.28 *Let $1/2 \leq a < 1$, $0 < b < 1$ and $0 < T \leq 1$. Suppose that $C(\cdot) \in \mathcal{P}_T^a$. Then there are constants c_5 and c_6 independent of a, b, T and $C(\cdot)$ such that*

$$\int_0^T t^{-b} \left\| \int_0^t e^{(t-s)\Delta} C(s)^3 ds \right\|_{H_1}^2 dt \leq c_5 T^{2a-1} \beta_a^4 \int_0^T s^{-b} \|C(s)\|_{H_1}^2 ds, \quad (3.74)$$

$$\int_0^T t^{-b} \left\| \int_0^t e^{(t-s)\Delta} C(s) \cdot \partial C(s) ds \right\|_{H_1}^2 dt \leq c_6 T^{a-\frac{1}{2}} \beta_a^2 \int_0^T s^{-b} \|C(s)\|_{H_1}^2 ds. \quad (3.75)$$

Proof. We are going to use Lemma 3.25 for each inequality, with the appropriate choice of p .

Take $p = p_a$ in (3.67) and choose $f(s) = C(s)^3$ in (3.68). In view of (3.71) and (3.73) we find

$$\begin{aligned} & \int_0^T t^{-b} \left\| \int_0^t e^{(t-s)\Delta} C(s)^3 ds \right\|_{H_1}^2 dt \\ & \leq T^{2a-1} \int_0^T s^{-b} \|C(s)^3\|_{p_a}^2 ds (\kappa_{p_a} e^T)^2 \\ & \leq T^{2a-1} \int_0^T s^{-b} \|C(s)\|_{H_1}^2 ds (c^2 \kappa_6 \beta_a^2)^2 (\kappa_{p_a} e^T)^2, \end{aligned}$$

which is (3.74) upon taking $c_5 = c^4 \kappa_6^2 \kappa_{p_a}^2 e^2$, since $T \leq 1$.

Take $p = r_a$ in (3.67) and choose $f(s) = C(s) \cdot \partial C(s)$ in (3.68). In view of (3.72) and (3.73) we find

$$\begin{aligned} & \int_0^T t^{-b} \left\| \int_0^t e^{(t-s)\Delta} C(s) \cdot \partial C(s) ds \right\|_{H_1}^2 dt \\ & \leq T^{a-(1/2)} \int_0^T s^{-b} \|C(s) \cdot \partial C(s)\|_{r_a}^2 ds \cdot (\kappa_{r_a} e^T)^2 \\ & \leq T^{a-(1/2)} \int_0^T s^{-b} \|C(s)\|_{H_1}^2 ds \cdot (c\beta_a)^2 (\kappa_{r_a} e^T)^2, \end{aligned}$$

which is (3.75) upon taking $c_6 = c^2 \kappa_{r_a}^2 e^2$. c_5 and c_6 can be taken independent of a because κ_p is bounded for $6/5 \leq p \leq 2$. ■

Lemma 3.29 *Let $1/2 \leq a < 1$, $0 < b < 1$ and $0 < T \leq 1$. There is a constant c_7 independent of a, b, T and $C(\cdot)$ such that*

$$\int_0^T t^{-b} \|w(t)\|_{H_1}^2 dt \leq c_7 \left(T^{a-\frac{1}{2}} \beta_a^2 \right) \left(T^{a-\frac{1}{2}} \beta_a^2 + 1 \right) \int_0^T s^{-b} \|C(s)\|_{H_1}^2 ds, \quad (3.76)$$

wherein $w(t)$ is defined by (3.31).

Proof. In view of the definition (3.31) we see that $\int_0^T t^{-b} \|w(t)\|_{H_1}^2 dt$ is at most twice the sum of the left hand sides of (3.74) and (3.75). Take $c_7 = 2 \max(c_5, c_6)$ and add (3.74) and (3.75) to arrive at (3.76). ■

Proof of Theorem 3.18. The integral equation (3.5) is given by $C(t) = e^{t\Delta} C_0 + w(t)$ in view of the definition (3.31) of w . Hence, for any number $b \in (0, 1)$, we have

$$\begin{aligned} & \left(\int_0^T t^{-b} \|C(t)\|_{H_1}^2 dt \right)^{1/2} \\ & \leq \left(\int_0^T t^{-b} \|e^{t\Delta} C_0\|_{H_1}^2 dt \right)^{1/2} + \left(\int_0^T t^{-b} \|w(t)\|_{H_1}^2 dt \right)^{1/2} \\ & \leq \left(\int_0^T t^{-b} \|e^{t\Delta} C_0\|_{H_1}^2 dt \right)^{1/2} + \mu_a \left(\int_0^T s^{-b} \|C(s)\|_{H_1}^2 ds \right)^{1/2}, \quad (3.77) \end{aligned}$$

where, by (3.76), we can take

$$\mu_a = \sqrt{c_7 \left(T^{a-\frac{1}{2}} \beta_a^2 \right) \left(T^{a-\frac{1}{2}} \beta_a^2 + 1 \right)}. \quad (3.78)$$

The rest of the proof hinges on whether we can take $\mu_a < 1$ by choosing T and/or C_0 suitably. Here the proof diverges into two cases. Either $a > 1/2$, in which case choosing T small makes μ_a small because T occurs with a strictly positive power in μ_a . Or else $a = 1/2$, in which case

$$\mu_{1/2}^2 = c_7 \beta_{1/2}^2 (\beta_{1/2}^2 + 1). \quad (3.79)$$

In this case we will show that choosing both $\|C_0\|_{H_{1/2}}$ small and T small ensures that $\mu_{1/2} < 1/2$.

Let us carry out the details for the case $a > 1/2$ first. We are considering a particular solution of the integral equation (3.5). Clearly β_a , defined in (3.70), decreases as T decreases and is finite for all T by Sobolev's inequality. Consequently μ_a decreases to zero as $T \downarrow 0$ because $a - 1/2 > 0$. Choose T so small that $\mu_a \leq 1/2$. We would like to subtract the last term on the right of (3.77) from the left side to obtain a bound on the b action for $b = a$. However we do not know that the right side is finite for $b = a$ since this is what we are trying to prove. Let $b < a$. Since $C(\cdot)$ lies in the path space \mathcal{P}_T^a we know that $\|C(s)\|_{H_1}^2 = o(s^{a-1})$ as $s \downarrow 0$ by (3.7). Hence $s^{-b}\|C(s)\|_{H_1}^2 = o(s^{a-b-1})$ as $s \downarrow 0$. Since $a - b - 1 > -1$ it follows that $\int_0^T s^{-b}\|C(s)\|_{H_1}^2 ds < \infty$. We can therefore subtract the last term in (3.77) from the left side to find, after squaring,

$$(1/4) \int_0^T t^{-b} \|C(t)\|_{H_1}^2 dt \leq \int_0^T t^{-b} \|e^{t\Delta} C_0\|_{H_1}^2 dt \quad \text{for } 0 < b < a. \quad (3.80)$$

As $b \uparrow a$ we have $t^{-b} \uparrow t^{-a}$ on $(0, 1]$. Therefore the monotone convergence theorem now shows that

$$(1/4) \int_0^T t^{-a} \|C(t)\|_{H_1}^2 dt \leq \int_0^T t^{-a} \|e^{t\Delta} C_0\|_{H_1}^2 dt \quad (3.81)$$

The right side is finite by (3.15). This completes the proof of Theorem 3.18 in case $a > 1/2$.

Suppose now that $a = 1/2$. By assumption, $C(\cdot)$ is a continuous function into $H_{1/2}$ and therefore, by Sobolev, a continuous function into L^3 . Hence, if $\|C_0\|_{H_{1/2}}$, and therefore $\|C_0\|_3$, are small then $\|C(s)\|_3$ remains small for a short time. Thus we can choose $\|C_0\|_3$ small and then choose $T > 0$ so small that $\beta_{1/2} \equiv \sup \|C(s)\|_3$ is small enough to ensure, by virtue of (3.79), that $\mu_{1/2} \leq 1/2$. The remaining details of the proof are the same as for the case $a > 1/2$. This completes the proof of Theorem 3.18 as well as Parts *ii*) and *iii*) of Theorem 3.3. ■

3.6 Mild solutions are strong solutions

We wish to show that a function $C(\cdot) \in \mathcal{P}_T^a$ is a solution to the integral equation (3.5) if and only if it is a strong solution to the differential equation (2.22). Combined with Theorem 3.18, this will complete the proof of Theorem 2.20.

Theorem 3.30 *Let $1/2 \leq a < 1$. Suppose that $C(\cdot)$ is a mild solution on $[0, T)$ lying in \mathcal{P}_T^a . Then $C(\cdot)$ is a strong solution on $(0, T)$. Moreover, $C(\cdot) \in C^\infty((0, T) \times M; \Lambda^1 \otimes \mathfrak{k})$ and satisfies, for $0 < t < T$, the Neumann, resp. Dirichlet boundary conditions*

$$(N) \ C(t)_{norm} = 0, \ (dC(t))_{norm} = 0, \ (B_{C(t)})_{norm} = 0. \quad (3.82)$$

$$(D) \ C(t)_{tan} = 0, \ (dC(t))_{tan} = 0, \ (B_{C(t)})_{tan} = 0, \ (d^*C(t))_{tan} = 0. \quad (3.83)$$

Two strong solutions lying in \mathcal{P}_T^a are equal.

Proof. Let $0 < \tau < t < T$. Suppose that $C(\cdot)$ is a solution of the integral equation (3.5) lying in \mathcal{P}_T^a . Define $f(s) = X(C(s))$. The integral equation (3.5) may be rewritten as

$$\begin{aligned} C(t) &= e^{t\Delta} C_0 + \int_0^t e^{(t-s)\Delta} f(s) ds \\ &= e^{(t-\tau)\Delta} \left(e^{\tau\Delta} C_0 + \int_0^\tau e^{(\tau-s)\Delta} f(s) ds \right) + \int_\tau^t e^{(t-s)\Delta} f(s) ds \\ &= e^{(t-\tau)\Delta} C(\tau) + \int_\tau^t e^{(t-s)\Delta} f(s) ds. \end{aligned} \quad (3.84)$$

Thus $C(t)$ is a mild solution over the interval $[\tau, T]$ with initial value $C(t)|_{t=\tau} = C(\tau)$. Now $C(\cdot)$ is a continuous function into H_1 over $[\tau, T]$ as well as into H_a and therefore lies in the path space $\mathcal{P}_{[\tau, T]}^a$, defined as in Notation 3.2 but with τ as the origin. The corresponding path space $\hat{\mathcal{P}}_{[\tau, T]}$ used in [2] is contained in $\mathcal{P}_{[\tau, T]}^a$. Since $C(\tau) \in H_1$, [2, Theorem 7.3] assures that there exists a mild solution $\hat{C}(\cdot)$ over some interval $[\tau, \tau + \epsilon]$ with initial value $C(\tau)$ and which lies in $\hat{\mathcal{P}}_{[\tau, \tau + \epsilon]}$. We may assume without loss of generality that $\tau + \epsilon < T$. But $\hat{\mathcal{P}}_{[\tau, \tau + \epsilon]} \subset \mathcal{P}_{[\tau, \tau + \epsilon]}^a$ and mild solutions are unique in $\mathcal{P}_{[\tau, \tau + \epsilon]}^a$. Hence $\hat{C}(t) = C(t)$ for $t \in [\tau, \tau + \epsilon]$. Now [2, Theorem 7.3] also assures that \hat{C} is a strong solution over $(\tau, \tau + \epsilon)$. Therefore $C(\cdot)$ is a strong solution over $(\tau, \tau + \epsilon)$. Since τ is arbitrary in $(0, T)$, $C(\cdot)$ is a strong solution

over $(0, T)$. The same argument, using again [2, Theorem 7.3], shows that $C(\cdot) \in C^\infty((0, T) \times M; \Lambda^1 \otimes \mathfrak{k})$. Moreover, by [2, Corollary 7.10], $\hat{C}(t)$ satisfies the boundary conditions (3.82), resp. (3.83) over $(\tau, \tau + \epsilon)$. Therefore $C(\cdot)$ does also.

Conversely, suppose that $C(\cdot)$ is a strong solution to the differential equation (2.22) and which satisfies the conditions a) and b) of Theorem 2.20. That is to say, $C(\cdot) \in \mathcal{P}_T^a$. If $0 < \tau < T$ then $C(\cdot)$ is a continuous function into H_1 over $[\tau, T]$. Consequently, as shown in [2, Proof of Theorem 2.13], $C(\cdot)$ is a solution to the integral equation (3.84) for $\tau \leq t \leq T$. Since, for fixed $t > 0$, the integral equation (3.84) holds for all $\tau \in (0, t)$, we can let $\tau \downarrow 0$ and find that (3.84) holds also for $\tau = 0$ by observing first, that $C(\cdot)$ is continuous into L^2 (in fact into H_a) over $[0, T]$, allowing the strong limit in the first term, and second, that the estimates on $f(s)$ made in Section 3.3 on the basis of the hypotheses of Theorem 2.20 allow us to take the limit in the integral term in (3.84). Hence a strong solution to the differential equation (2.22) which lies in \mathcal{P}_T^a is a solution to the integral equation (3.5). Uniqueness for such strong solutions now follows.

This completes the existence and uniqueness portion of Theorem 2.20. Since a strong solution is also a mild solution we can apply Theorem 3.18 to deduce the remaining, finite action, assertions of Theorem 2.20. ■

4 Initial behavior of solutions to the augmented equation

We are going to derive energy estimates for the first and second order spatial derivatives of a solution $C(\cdot)$ to the augmented Yang-Mills heat equation (2.22) and then use these to derive additional bounds via the method of Neumann domination. These estimates will be used in Section 6 to establish the properties of the conversion group which are needed to recover the desired solution $A(\cdot)$ of (2.5) from $C(\cdot)$.

In this section we will take M to be either all of \mathbb{R}^3 or the closure of a bounded, convex, open subset of \mathbb{R}^3 with smooth boundary.

The main technique in the next few subsections will be based on the Gaffney-Friedrichs-Sobolev inequality, which asserts, for our convex subset M of \mathbb{R}^3 , that for any integer $p \geq 1$ and any \mathfrak{k} valued p -form ω (satisfying

appropriate boundary conditions) there holds

$$\|\omega\|_6^2 \leq \kappa^2 \left\{ \|d_C^* \omega\|_2^2 + \|d_C \omega\|_2^2 + \lambda(B_C) \|\omega\|_2^2 \right\} \quad (4.1)$$

for any \mathfrak{k} valued connection form C on M with curvature B_C . Here we have written

$$\lambda(B) = 1 + \gamma \|B\|_2^4, \quad (4.2)$$

where $\gamma \equiv (27/4)\kappa^6 c^4$ is a constant depending only on a Sobolev constant κ for M , which can be \mathbb{R}^3 , and the commutator bound c defined in Section 3.3. See [2, Theorem 2.17, Remark 2.18 and Equ.(4.31)] for the derivation of the inequality (4.1). If M is not convex then the inequality (4.1) still holds, but with different constants in (4.2) provided that the second fundamental form of ∂M is bounded below.

Gaffney-Friedrichs inequalities [9, 25, 27, 26] give information about the gradient of a form in terms of the exterior derivative and co-derivative of the form. The use of these is essential for us because the differential equations are posed in terms of the gauge covariant exterior derivative d_C and its co-derivative d_C^* , whereas Sobolev inequalities require information about gradients.

For a \mathfrak{k} -valued 0-form ϕ on M one has the simple gauge invariant Sobolev inequality

$$\|\phi\|_6^2 \leq \kappa_6^2 \left(\|d_C \phi\|_2^2 + \|\phi\|_2^2 \right) \quad (4.3)$$

where $\kappa_6 \leq \kappa$. This is valid independently of boundary conditions on ϕ . It is just a consequence of Kato's inequality, $|\text{grad } |\phi(x)|| \leq |d_{C(x)} \phi(x)|$. If $M = \mathbb{R}^3$ then the summand $\|\phi\|_2^2$ is not needed. See, e.g., [2, Notation 2.16] for further discussion.

4.1 Identities

Suppose that $C(\cdot)$ is a solution to (2.22) over some interval. Define

$$\phi(s) = d^* C(s). \quad (4.4)$$

We are going to derive energy estimates for ϕ and B_C and their gauge covariant derivatives. Similar energy estimates have been made for the curvature of $A(\cdot)$ and its covariant derivatives in [2] and [3]. The augmented equation (2.22) is a little more complicated than the Yang-Mills heat equation (2.5), which was the basis for the energy estimates in [2] and [3]. This reflects itself in slightly more complicated energy estimates for ϕ and B_C .

Lemma 4.1 (*Pointwise identities*) Suppose that $C(\cdot)$ is a smooth solution to (2.22) over $(0, T)$. Then

$$d\phi(s)/ds = d_C^* C' - [C \lrcorner C'], \quad (4.5)$$

$$= -d_C^* d_C \phi - [C \lrcorner C'] \quad \text{and} \quad (4.6)$$

$$= \Delta \phi + [C \lrcorner (d_C^* B_C - d\phi)]. \quad (4.7)$$

Further,

$$dB_C(s)/ds = \sum_{j=1}^3 (\nabla_j^C)^2 B_C + B_C \# B_C - [B_C, \phi], \quad (4.8)$$

where $\#$ denotes a pointwise product coming from the Bochner-Weitzenboch formula.

Proof. The definition (4.4) gives

$$d\phi/ds = (d/ds)d^* C = d^* C' = d_C^* C' - [C \lrcorner C'],$$

which proves (4.5). In view of the differential equation (2.22) we have $d_C^* C' = -d_C^* (d_C^* B_C + d_C \phi) = -d_C^* d_C \phi$ by Bianchi's identity. This proves (4.6). Expand $d_C^* d_C \phi = d^* (d\phi + [C, \phi]) + [C \lrcorner d_C \phi] = d^* d\phi + d^* [C, \phi] + [C \lrcorner d_C \phi]$ and use the identity $d^* [C, \phi] = [d^* C, \phi] + [C \lrcorner d\phi] = [C \lrcorner d\phi]$. The last equality follows from $[d^* C, \phi] = [\phi, \phi] = 0$. Thus $d_C^* d_C \phi = d^* d\phi + [C \lrcorner (d\phi + d_C \phi)]$. But $-[C \lrcorner C'] = [C \lrcorner (d_C^* B_C + d_C \phi)]$ by (2.22). Combine the last two equalities with (4.6) to find (4.7).

Over our flat manifold M the Bochner-Weitzenboch formula asserts that, for any \mathfrak{k} -valued p -form ω ,

$$-(d_C d_C^* + d_C^* d_C)\omega = \sum_{j=1}^3 (\nabla_j^C)^2 \omega + B_C \# \omega \quad (4.9)$$

for some pointwise product $\#$. Since $d_C B_C = 0$ by the Bianchi identity, we have

$$\begin{aligned} (d/ds)B_C(s) &= d_C C'(s) \\ &= -d_C (d_C^* B_C + d_C \phi) \\ &= -d_C d_C^* B_C - [B_C, \phi] \\ &= -(d_C d_C^* + d_C^* d_C)B_C - [B_C, \phi]. \end{aligned}$$

Insert (4.9) with $\omega = B_C$ to arrive at (4.8). ■

Lemma 4.2 (*Integral identities*) Suppose that $C(\cdot)$ is a smooth solution to (2.22) over $(0, T)$. Then

$$\frac{d}{ds} \left(\|B_C(s)\|_2^2 + \|\phi(s)\|_2^2 \right) + 2\|C'(s)\|_2^2 = -2(C', [C, \phi]) \quad (4.10)$$

and

$$\begin{aligned} \frac{d}{ds} \|C'(s)\|_2^2 + 2 \left\{ \|d_{C(s)} C'(s)\|_2^2 + \|d_{C(s)}^* C'(s)\|_2^2 \right\} \\ = -2(B_C, [C' \wedge C']) + 2([C \lrcorner C'], d_C^* C'). \end{aligned} \quad (4.11)$$

Proof. The identity (4.10) follows from the computation

$$\begin{aligned} (1/2) \frac{d}{ds} \left(\|B_C(s)\|_2^2 + \|\phi(s)\|_2^2 \right) &= (d_C C', B_C) + (\phi', \phi) \\ &= (C', d_C^* B_C) + (d_C^* C' - [C \lrcorner C'], \phi) \\ &= (C', d_C^* B_C) + (C', d_C \phi) - (C', [C, \phi]) \\ &= -\|C'\|_2^2 - (C', [C, \phi]). \end{aligned}$$

To prove (4.11) observe that

$$\begin{aligned} -C'' &= (d/ds)(d_C^* B_C + d_C \phi) \\ &= \left\{ d_C^* B'_C + [C' \lrcorner B_C] \right\} + \left\{ d_C \phi' + [C', \phi] \right\} \\ &= \left\{ d_C^* d_C C' + [C' \lrcorner B_C] \right\} + \left\{ d_C d_C^* C' - d_C [C \lrcorner C'] + [C', \phi] \right\}. \end{aligned}$$

Hence

$$\begin{aligned} (1/2)(d/ds) \|C'(s)\|_2^2 &= (C'', C') \\ &= - \left(\left\{ d_C^* d_C C' + [C' \lrcorner B_C] \right\} + \left\{ d_C d_C^* C' - d_C [C \lrcorner C'] + [C', \phi] \right\}, C' \right) \\ &= -\|d_C C'\|_2^2 - ([C' \lrcorner B_C], C') - \|d_C^* C'\|_2^2 + ([C \lrcorner C'], d_C^* C') - ([C', \phi], C') \\ &= -\|d_C C'\|_2^2 - \|d_C^* C'\|_2^2 - (B_C, [C' \wedge C']) + ([C \lrcorner C'], d_C^* C') \end{aligned}$$

because $([C', \phi], C') = (\phi, [C' \lrcorner C']) = 0$. This proves (4.11). ■

Remark 4.3 (Strategy) Typically, parabolic equations lead to energy decay via identities such as (4.10) and (4.11) when the right hand sides are small or easily controllable. However the non-linearities of the augmented equation

(2.22) produce strong terms on the right side with uncontrolled sign. We will balance out some of the strong terms on the right against half of the positive terms on the left. For this we will need L^6 bounds on some factors to estimate the non-linear terms on the right. These in turn will be obtained by applying the Gaffney-Friedrichs-Sobolev inequality to the next higher derivative.

4.2 Differential inequalities and initial behavior

The identities of the preceding subsection give the following differential inequalities with the help of the Gaffney-Friedrichs-Sobolev inequality. At the end of this subsection it will be shown how these differential inequalities give information about the initial behavior.

Theorem 4.4 *Suppose that $C(\cdot)$ is a strong solution to (2.22) on some interval. There are constants a_j, b_j depending only on Sobolev constants and the commutator norm c such that*

$$\frac{d}{ds} \left(\|B_C(s)\|_2^2 + \|\phi(s)\|_2^2 \right) + \|C'(s)\|_2^2 \leq \alpha(s) \|\phi(s)\|_2^2, \quad \text{Order1} \quad (4.12)$$

and

$$\frac{d}{ds} \|C'(s)\|_2^2 + \left(\|d_C^* C'(s)\|_2^2 + \|d_C C'(s)\|_2^2 \right) \leq \beta(s) \|C'(s)\|_2^2, \quad \text{Order2}, \quad (4.13)$$

where

$$\alpha(s) = a_1 + a_2 \|C(s)\|_6^4 \quad \text{and} \quad (4.14)$$

$$\beta(s) = b_1 + b_2 \|C(s)\|_6^4 + b_3 \|B_C(s)\|_2^4. \quad (4.15)$$

The constants are given explicitly in (4.19) and (4.27).

The proof depends on the following lemmas, in which it is assumed that $C(\cdot)$ is a strong solution to (2.22).

Lemma 4.5 (Order 1) *There are constants a_1, a_2 such that at each time s there holds*

$$2\| [C, \phi] \|_2^2 \leq \alpha(s) \|\phi\|_2^2 + (1/2) \|C'\|_2^2 \quad (4.16)$$

with $\alpha(s)$ given by (4.14).

Proof. Since $(d_C^*)^2 B_C = 0$, it follows that $d_C^* B_C$ and $d_C \phi$ are mutually orthogonal in L^2 . Consequently

$$\|C'\|_2^2 = \|d_C^* B_C\|_2^2 + \|d_C \phi\|_2^2. \quad (4.17)$$

Hence $\|d_C \phi\|_2^2 \leq \|C'(s)\|_2^2$. Sobolev's inequality (4.3) then gives, at each time s ,

$$(\kappa_6^{-1} \|\phi\|_6)^2 \leq \|C'\|_2^2 + \|\phi\|_2^2. \quad (4.18)$$

Therefore

$$\begin{aligned} 2\| [C, \phi] \|_2^2 &\leq 2c^2 \|C\|_6^2 \|\phi\|_3^2 \\ &\leq 2c^2 \|C\|_6^2 \|\phi\|_2 \|\phi\|_6 = \left(2\kappa_6 c^2 \|C\|_6^2 \|\phi\|_2\right) \left(\kappa_6^{-1} \|\phi\|_6\right) \\ &\leq (1/2)(2\kappa_6 c^2 \|C\|_6^2 \|\phi\|_2)^2 + (1/2)(\kappa_6^{-1} \|\phi\|_6)^2 \\ &\leq \left(2\kappa_6^2 c^4 \|C\|_6^4\right) \|\phi\|_2^2 + (1/2)(\|C'\|_2^2 + \|\phi\|_2^2), \end{aligned}$$

which is (4.16) with

$$a_1 = 1/2, \quad a_2 = 2\kappa_6^2 c^4. \quad (4.19)$$

■

Lemma 4.6 (*Order 2*)

$$\begin{aligned} 2\left|(B_C, [C' \wedge C'])\right| &\leq (1/2)\left\{\|d_C^* C'\|_2^2 + \|d_C C'\|_2^2\right\} \\ &\quad + (1/2)\left\{\lambda(B_C) + (3\kappa^2)^3 c^4 \|B_C\|_2^4\right\} \|C'\|_2^2. \end{aligned} \quad (4.20)$$

Proof. By the interpolation $\|f\|_4 \leq \|f\|_2^{1/4} \|f\|_6^{3/4}$ we have, for any $\eta > 0$,

$$\begin{aligned} 2\left|(B_C, [C' \wedge C'])\right| &\leq 2c \|B_C\|_2 \|C'\|_4^2 \\ &\leq 2c \|B_C\|_2 \|C'\|_2^{1/2} \|C'\|_6^{3/2} = \left(2\eta c \|B_C\|_2 \|C'\|_2^{1/2}\right) \left(\eta^{-1} \|C'\|_6^{3/2}\right) \\ &\leq (1/4) \left(2\eta c \|B_C\|_2 \|C'\|_2^{1/2}\right)^4 + (3/4) \left(\eta^{-1} \|C'\|_6^{3/2}\right)^{4/3} \\ &= (1/4) \left(2\eta c \|B_C\|_2\right)^4 \|C'\|_2^2 + (3/4) \eta^{-4/3} \|C'\|_6^2 \\ &= \left((1/2)(3\kappa^2)^3 c^4 \|B_C\|_2^4\right) \|C'\|_2^2 + \frac{1}{2\kappa^2} \|C'\|_6^2, \end{aligned} \quad (4.21)$$

wherein we have chosen $(2\eta)^4 = 2(3\kappa^2)^3$, which makes $(3/4)\eta^{-4/3} = 1/(2\kappa^2)$. By the Gaffney-Friedrichs-Sobolev inequality (4.1) with $\omega = C'(s)$ we have

$$\kappa^{-2}\|C'(s)\|_6^2 \leq \left\{ \|d_C^* C'\|_2^2 + \|d_C C'\|_2^2 + \lambda(B_C)\|C'\|_2^2 \right\}. \quad (4.22)$$

Insert (4.22) into the last term in (4.21) to arrive at (4.20). ■

Lemma 4.7 (*Order 2*)

$$\begin{aligned} 2\left|([C \lrcorner C'], d_C^* C')\right| &\leq (1/2)\left\{ \|d_C^* C'\|_2^2 + \|d_C C'\|_2^2 \right\} \\ &\quad + \left\{ \lambda(B_C)/4 + (4\kappa c^2)^2 \|C\|_6^4 \right\} \|C'\|_2^2. \end{aligned} \quad (4.23)$$

Proof. Hölder's inequality and the interpolation $\|f\|_3 \leq \|f\|_2^{1/2} \|f\|_6^{1/2}$ give

$$\begin{aligned} 4\| [C \lrcorner C'] \|_2^2 &\leq 4c^2 \|C\|_6^2 \|C'\|_3^2 \\ &\leq 4c^2 \|C\|_6^2 \|C'\|_2 \|C'\|_6 = \left(4\kappa c^2 \|C\|_6^2 \|C'\|_2 \right) \left(\kappa^{-1} \|C'\|_6 \right) \\ &\leq \left(4\kappa c^2 \|C\|_6^2 \|C'\|_2 \right)^2 + \frac{1}{4\kappa^2} \|C'\|_6^2. \end{aligned}$$

Hence,

$$\begin{aligned} 2\left|([C \lrcorner C'], d_C^* C')\right| &\leq (1/4)\|d_C^* C'\|_2^2 + 4\| [C \lrcorner C'] \|_2^2 \\ &\leq (1/4)\|d_C^* C'\|_2^2 + \left(4\kappa c^2 \|C\|_6^2 \|C'\|_2 \right)^2 + \frac{1}{4\kappa^2} \|C'\|_6^2. \end{aligned} \quad (4.24)$$

Insert the Gaffney-Friedrichs-Sobolev inequality (4.22) into the last term to arrive at (4.23). ■

Proof of Theorem 4.4. To prove (4.12) use the estimate in (4.16) to find

$$\begin{aligned} \left| 2(C', [C, \phi]) \right| &\leq (1/2)\|C'\|_2^2 + 2\| [C, \phi] \|_2^2 \\ &\leq \|C'\|_2^2 + \alpha(s)\|\phi\|_2^2. \end{aligned}$$

Estimate the right side of the identity (4.10) by this bound and then cancel the term $\|C'\|_2^2$ with part of the left side of (4.10) to arrive at (4.12).

To prove (4.13) add the inequalities (4.20) and (4.23) to find

$$\begin{aligned} & \left| 2(B_C, [C' \wedge C']) + 2([C \sqcup C'], d_C^* C') \right| \\ & \leq \left\{ \|d_C^* C'\|_2^2 + \|d_C C'\|_2^2 \right\} + \beta(s) \|C'(s)\|_2^2, \end{aligned} \quad (4.25)$$

where

$$\begin{aligned} \beta(s) &= (1/2) \left\{ \lambda(B_C) + (3\kappa^2)^3 c^4 \|B_C\|_2^4 \right\} + \left\{ \lambda(B_C)/4 + (4\kappa c^2)^2 \|C\|_6^4 \right\} \\ &= (3/4) + \left((\gamma/2) + 27\kappa^6 c^4 + (\gamma/4) \right) \|B_C\|_2^4 + (4\kappa c^2)^2 \|C\|_6^4. \end{aligned} \quad (4.26)$$

We can now estimate the right side of (4.11) using (4.25). We see that there is partial cancelation of the expression in braces in (4.11), leaving exactly (4.13) with $\beta(s)$ given by (4.26). From this we can compute the coefficients in (4.15) to be

$$b_1 = 3/4, \quad b_2 = (4\kappa c^2)^2, \quad b_3 = (\kappa^6 c^4) b_0 \quad (4.27)$$

with $b_0 = 19 \cdot 27/16$. ■

The differential inequalities of Theorem 4.4 will yield information about the initial behavior of $C(t)$ and its derivatives with the help of the next elementary lemma. We will apply it several times in the following sections.

Lemma 4.8 (*Initial behavior from differential inequalities*) Suppose that f, g, h are nonnegative continuous functions on $(0, t]$ and that f is differentiable. Suppose also that

$$(d/ds)f(s) + g(s) \leq h(s), \quad 0 < s \leq t. \quad (4.28)$$

Let $-\infty < b < 1$ and assume that

$$\int_0^t s^{-b} f(s) ds < \infty. \quad (4.29)$$

Then

$$t^{1-b} f(t) + \int_0^t s^{(1-b)} g(s) ds \leq \int_0^t s^{(1-b)} h(s) ds + (1-b) \int_0^t s^{-b} f(s) ds. \quad (4.30)$$

If equality holds in (4.28) then equality holds in (4.30).

Suppose, instead of (4.28), that $f' \leq 0$. Then

$$t^{1-b}f(t) + \int_0^t s^{1-b}(-f'(s))ds = (1-b) \int_0^t s^{-b}f(s)ds \quad (4.31)$$

and, for $b \in [0, 1)$,

$$(1-b) \int_0^t f(s)^q ds \leq \left\{ (1-b) \int_0^t s^{-b}f(s)ds \right\}^q \quad \text{if } q = (1-b)^{-1}. \quad (4.32)$$

In particular,

$$(1/2) \int_0^t f(s)^2 ds \leq \left\{ (1/2) \int_0^t s^{-1/2}f(s)ds \right\}^2. \quad (4.33)$$

Proof. Multiply (4.28) by s^{-b} to find

$$(d/ds)s^{-b}f(s) + bs^{-b-1}f(s) + s^{-b}g(s) \leq s^{-b}h(s).$$

For $0 < \sigma \leq t$ integrate this inequality from σ to t to arrive at

$$t^{-b}f(t) + b \int_\sigma^t s^{-b-1}f(s)ds + \int_\sigma^t s^{-b}g(s)ds \leq \int_\sigma^t s^{-b}h(s)ds + \sigma^{-b}f(\sigma).$$

Integrate this inequality now with respect to σ over the interval $[0, t]$. Since all integrands are positive we can reverse the order of integration in the three double integrals. This just results in multiplying each integrand by s , giving

$$t^{1-b}f(t) + b \int_0^t s^{-b}f(s)ds + \int_0^t s^{1-b}g(s)ds \leq \int_0^t s^{1-b}h(s)ds + \int_0^t \sigma^{-b}f(\sigma)d\sigma.$$

Subtract the second term on the left from the second term on the right to arrive at (4.30). If equality holds in (4.28) then equality holds in all steps.

To prove (4.31) take $g = -f'$ and $h = 0$ in (4.28). Then equality holds in (4.28) and (4.30) reduces to (4.31).

To prove (4.32) let $\rho(\sigma) = (1-b) \int_0^\sigma s^{-b}f(s)ds$ for $0 \leq \sigma \leq t$. (4.31) clearly holds when t is replaced by σ and therefore $\sigma^{1-b}f(\sigma) \leq \rho(\sigma)$ for $0 < \sigma \leq t$. Consequently, since $(1-b)(q-1) = b$ and $q \geq 1$, we have

$$\begin{aligned} \int_0^t f(s)^q ds &= \int_0^t \left(s^{1-b}f(s) \right)^{q-1} s^{-b}f(s)ds \\ &\leq \left(\sup_{0 < s \leq t} s^{1-b}f(s) \right)^{q-1} \int_0^t s^{-b}f(s)ds \\ &\leq \rho(t)^{q-1} \rho(t) / (1-b) \end{aligned}$$

by the monotonicity of $\rho(\cdot)$. This proves (4.32). Choose $b = 1/2$, and consequently $q = 2$, in (4.32) to arrive at (4.33). ■

Remark 4.9 A seemingly shorter proof of (4.30) can be derived by simply using the identity

$$(d/ds)s^{1-b}f(s) + s^{1-b}g(s) \leq s^{1-b}h(s) + (1-b)s^{-b}f(s), \quad (4.34)$$

which follows from (4.28), and then integrating it over $(0, t]$. However the integrated term at the lower limit is $\lim_{s \downarrow 0} s^{1-b}f(s)$, and we have no way of knowing in advance that this limit is zero, except in some special circumstances. For example in the special case $f' \leq 0$, leading to (4.31), and for $0 < b < 1$, the monotonicity of f gives $s^{1-b}f(s) = f(s)(1-b) \int_0^s \sigma^{-b} d\sigma \leq (1-b) \int_0^s \sigma^{-b} f(\sigma) d\sigma \rightarrow 0$ as $s \downarrow 0$ because the integrand is integrable.

4.3 Initial behavior, order 0

In the classical integrating factor method for solving an ordinary differential equation such as $dx/dt = f(t)x(t) + g(t)$, one changes the “dependent variable” to $y(t) \equiv e^{-\int_0^t f(s)ds}x(t)$ and then uses the equivalent equation $dy/dt = e^{-\int_0^t f(s)ds}g(t)$ to solve for $y(t)$ as an integral. The inequalities (4.12) and (4.13) lend themselves to just such a use of integrating factors $e^{-\int_0^t \alpha(s)ds}$ and $e^{-\int_0^t \beta(s)ds}$ respectively. However both functions α and β are quite singular near $s = 0$. In fact from the sole knowledge that $C(\cdot)$ lies in $\mathcal{P}_T^{1/2}$ one can only deduce that each function is no worse than $o(s^{-1})$ near $s = 0$. The existence of $\int_0^t \alpha(s)ds$, and therefore its utility as an integrating factor, is thus in question in the critical case, $a = 1/2$. The same is the case with $\beta(s)$. However it was shown in Theorem 3.18 that for small initial data the solution has finite strong action when $a = 1/2$. Here it will be shown that if $C(\cdot)$ has finite strong action then α and β are integrable over $(0, t]$. Their use in the method of integrating factors will then give detailed information about the initial behavior of the various derivatives of $C(\cdot)$ of interest to us. This differs significantly from the non-critical case $a > 1/2$, where finite strong a -action is automatic. If $a > 1/2$ one need not assume that $\|C_0\|_{H_a}$ is small in order to use these integrating factors.

However even when $C(\cdot)$ has infinite action many of the qualitative conclusions needed for the recovery of A from C hold. This will be shown in Section 7.

The main theorems of this and the next two subsections concern the initial behavior of solutions to the augmented Yang-Mills equation (2.22). Some simple aspects of this behavior are just consequences of the fact that the function $C(\cdot)$ lies in the path space \mathcal{P}_T^a . It need not be a solution. The following theorem (which we have labeled Order 0) lists some of these properties.

Notation 4.10 Denote by α and β the functions defined in (4.14) and (4.15). Define

$$\alpha_s^t = \int_s^t \alpha(\sigma) d\sigma, \quad \beta_s^t = \int_s^t \beta(\sigma) d\sigma \quad \text{for } 0 \leq s \leq t \leq T < \infty. \quad (4.35)$$

Theorem 4.11 (Order 0) Let $1/2 \leq a < 1$ and $0 < T < \infty$. Suppose that $C(\cdot)$ lies in the path space \mathcal{P}_T^a . Then

$$s^{1-a} \left(\|\phi(s)\|_2^2 + \|B_C(s)\|_2^2 \right) \text{ and } s^{2-2a} \|C(s)\|_6^4 \text{ are bounded on } (0, T] \quad (4.36)$$

and go to zero as $s \downarrow 0$. Further,

$$\sup_{0 < s < T} s^{2-2a} \left(\|B_C(s)\|_2^4 + \|\phi(s)\|_2^4 + \lambda(B_C(s)) \right) < \infty, \quad (4.37)$$

$$\alpha_\infty \equiv \sup_{0 < s \leq T} s^{2-2a} \alpha(s) < \infty, \quad \beta_\infty \equiv \sup_{0 < s \leq T} s^{2-2a} \beta(s) < \infty. \quad (4.38)$$

In particular, if $a = 1/2$, then

$$s^{1/4} \|B_C(s)\|_2, \quad s^{1/4} \|\phi(s)\|_2 \text{ and } s \|C(s)\|_6^4 \text{ are bounded on } (0, T] \quad (4.39)$$

and go to zero as $s \downarrow 0$. Further,

$$\sup_{0 < s < T} s \left(\|B_C(s)\|_2^4 + \|\phi(s)\|_2^4 \right) < \infty, \quad \sup_{0 < s < T} s \lambda(B_C(s)) < \infty, \quad (4.40)$$

$$\alpha_\infty \equiv \sup_{0 < s \leq T} s \alpha(s) < \infty, \quad \text{and} \quad \beta_\infty \equiv \sup_{0 < s \leq T} s \beta(s) < \infty. \quad (4.41)$$

If $C(\cdot) \in \mathcal{P}_T^a$ and in addition has finite strong a -action then

$$\int_0^T \|C(s)\|_{H_1}^4 ds < \infty, \quad (4.42)$$

$$\int_0^T s^{-a} \|B_C(s)\|_2^2 ds < \infty \quad \text{and} \quad \int_0^T s^{-a} \|\phi(s)\|_2^2 ds < \infty. \quad (4.43)$$

Further,

$$\int_0^T \left(\|B_C(s)\|_2^4 + \|\phi(s)\|_2^4 + \|C(s)\|_6^4 + \lambda(B_C(s)) \right) ds < \infty, \quad (4.44)$$

$$\alpha_0^T < \infty, \quad \text{and} \quad \beta_0^T < \infty. \quad (4.45)$$

In particular (4.42)- (4.45) hold if $C(\cdot) \in \mathcal{P}_T^a$ with $1/2 < a < 1$, and also hold for $a = 1/2$ if $C(\cdot) \in \mathcal{P}_T^{1/2}$ and $\|C_0\|_{H_{1/2}}$ is sufficiently small.

Proof. The curvature B_C is given by the usual expression $B_C = dC + C \wedge C$ for any connection form C . We can bound the L^2 norm of B_C as follows.

$$\begin{aligned} \|B_C\|_2 &\leq \|dC\|_2 + \|C \wedge C\|_2 \\ &\leq \|C\|_{H_1} + c\|C\|_3\|C\|_6 \\ &\leq \|C\|_{H_1} + c\kappa_6\|C\|_3\|C\|_{H_1}. \end{aligned} \quad (4.46)$$

Thus

$$\begin{aligned} \|B_C(s)\|_2 &\leq \|C(s)\|_{H_1}(1 + c\kappa_6\|C(s)\|_3) \\ &\leq c_3\|C(s)\|_{H_1}, \end{aligned} \quad (4.47)$$

where

$$c_3 = 1 + c\kappa_6 \sup_{0 < s \leq T} \|C(s)\|_3. \quad (4.48)$$

The constant c_3 is finite because $C(\cdot)$ lies in $\mathcal{P}_T^a \subset \mathcal{P}_T^{1/2}$ and is therefore a continuous function on $[0, T]$ into $H_{1/2}(M)$ and therefore into $L^3(M)$. Now $s^{1-a}\|C(s)\|_{H_1}^2$ is bounded on $(0, T)$ because $C(\cdot) \in \mathcal{P}_T^a$. Hence $s^{1-a}\|B_C(s)\|_2^2$ is bounded on $(0, T]$. Since

$$\|\phi(s)\|_2 \leq \|C(s)\|_{H_1} \quad \text{and} \quad \|C(s)\|_6 \leq \kappa_6\|C(s)\|_{H_1} \quad (4.49)$$

both of these functions are also bounded after multiplying by $s^{(1-a)/2}$. This completes the proof of (4.36). The inequalities (4.37) and (4.38) now follow from (4.36) in view of the definitions (4.2), (4.14) and (4.15). Put $a = 1/2$ to derive the special case (4.39) - (4.41).

Assume now that $C(\cdot)$ has finite strong a -action. Since $s^{1-a}\|C(s)\|_{H_1}^2$ is bounded on $(0, T]$ and $a \geq 1/2$ we have

$$\begin{aligned} \int_0^T \|C(s)\|_{H_1}^4 ds &= \int_0^T s^{2a-1} \left(s^{1-a} \|C(s)\|_{H_1}^2 \right) \left(s^{-a} \|C(s)\|_{H_1}^2 \right) ds \\ &\leq T^{2a-1} \sup_{0 < s < T} \left(s^{1-a} \|C(s)\|_{H_1}^2 \right) \int_0^T \left(s^{-a} \|C(s)\|_{H_1}^2 \right) ds \\ &< \infty. \end{aligned}$$

This proves (4.42). By (4.47) we have

$$\int_0^T s^{-a} \|B_C(s)\|_2^2 ds \leq c_3^2 \int_0^T s^{-a} \|C(s)\|_{H_1}^2 ds, \quad (4.50)$$

which is finite by the assumption of finite strong a -action. Since $\|\phi(s)\|_2 \leq \|C(s)\|_{H_1}$ the second integral in (4.43) is also finite. This proves (4.43).

The first three terms in the integral in (4.44) are dominated by a constant times $\|C(s)\|_{H_1}^4$, by (4.47) and (4.49). Hence the integral of these terms is finite. So is the integral of the last term, by the definition (4.2). The inequalities in (4.45) now follow from the definitions (4.14), (4.15) and (4.35). \blacksquare

4.4 Initial behavior, order 1

Theorem 4.12 (*First order energy estimate*) *Let $1/2 \leq a < 1$. Suppose that $C(\cdot)$ is a strong solution to the augmented equation (2.22) lying in \mathcal{P}_T^a and having finite strong a -action, i.e., (2.25) holds. Then*

$$\begin{aligned} t^{1-a} \left(\|B_C(t)\|_2^2 + \|\phi(t)\|_2^2 \right) + \int_0^t s^{1-a} e^{\alpha_s^t} \|C'(s)\|_2^2 ds &\quad (\text{Order 1}) \\ &\leq (1-a) \int_0^t s^{-a} e^{\alpha_s^t} \left(\|B_C(s)\|_2^2 + \|\phi(s)\|_2^2 \right) ds < \infty. \end{aligned} \quad (4.51)$$

The following weighted L^6 bound holds.

$$\int_0^T s^{1-a} \left(\|B_C(s)\|_6^2 + \|\phi(s)\|_6^2 \right) ds < \infty. \quad (\text{Order 1}) \quad (4.52)$$

Furthermore,

$$\int_0^T s^{1-a} \left(\|d_C^* B_C(s)\|_2^2 + \|d_C \phi(s)\|_2^2 + \|d\phi(s)\|_2^2 \right) ds < \infty. \quad (\text{Order 1}) \quad (4.53)$$

Proof. By hypothesis $C(\cdot)$ lies in \mathcal{P}_T^a and has finite strong a-action. Therefore $\alpha_0^T < \infty$ by (4.45) of Theorem 4.11. We will use this for the following bounds.

Let $\zeta(t) = \alpha_0^t$ and define $u(s) = \|B_C(s)\|_2^2 + \|\phi(s)\|_2^2$. The inequality (4.12) shows that $u'(s) + \|C'(s)\|_2^2 \leq \zeta'(s)u(s)$. Hence

$$\frac{d}{ds} \left(e^{-\zeta(s)} u(s) \right) + e^{-\zeta(s)} \|C'(s)\|_2^2 \leq 0. \quad (4.54)$$

Let $f(s) = e^{-\zeta(s)} u(s)$, $g(s) = e^{-\zeta(s)} \|C'(s)\|_2^2$ and $h(s) = 0$. We will apply Lemma 4.8 with these choices of f, g and h and with the choice $b = a$. Then (4.30) asserts that

$$t^{1-a} e^{-\zeta(t)} u(t) + \int_0^t s^{1-a} e^{-\zeta(s)} \|C'(s)\|_2^2 ds \leq (1-a) \int_0^t s^{-a} e^{-\zeta(s)} u(s) ds.$$

Multiply by $e^{\zeta(t)}$ to arrive at (4.51). Since $\alpha_0^T < \infty$ by (4.45), the inequalities in (4.43) show that the right hand side of (4.51) is finite.

To prove the weighted L^6 bound (4.52) we can use Sobolev's inequality (4.3) and the Gaffney-Friedrichs-Sobolev inequality (4.1), which give, respectively, (since $\kappa_6^2 \leq \kappa^2$)

$$\|\phi(s)\|_6^2 \leq \kappa^2 (\|d_C \phi(s)\|_2^2 + \|\phi(s)\|_2^2) \quad (4.55)$$

$$\|B_C(s)\|_6^2 \leq \kappa^2 (\|d_C^* B_C(s)\|_2^2 + \lambda(B_C(s)) \|B_C(s)\|_2^2) \quad (4.56)$$

since $d_C B_C = 0$ by Bianchi's identity. Adding (4.55) and (4.56), and using the identity (4.17), we find

$$\begin{aligned} \kappa^{-2} \left(\|B_C(s)\|_6^2 + \|\phi(s)\|_6^2 \right) \\ \leq \|C'(s)\|_2^2 + \|\phi(s)\|_2^2 + \lambda(B_C(s)) \|B_C(s)\|_2^2. \end{aligned} \quad (4.57)$$

Multiply (4.57) by s^{1-a} to find

$$\begin{aligned} \kappa^{-2} s^{1-a} \left(\|B_C(s)\|_6^2 + \|\phi(s)\|_6^2 \right) \\ \leq s^{1-a} \|C'(s)\|_2^2 + s^{1-a} \|\phi(s)\|_2^2 + \lambda(B_C(s)) \{s^{1-a} \|B_C(s)\|_2^2\}. \end{aligned} \quad (4.58)$$

The first term on the right, $s^{1-a} \|C'(s)\|_2^2$, is integrable over $(0, T]$ by (4.51). The second term is bounded by (4.36) and therefore integrable. The third

term is an integrable function times a bounded function by (4.44) and (4.36), respectively. This proves (4.52).

Concerning the inequality (4.53) observe first that the integrability of the first two terms in (4.53) follows from the orthogonality relation (4.17) along with the relation $\int_0^T s^{1-a} \|C'(s)\|_2^2 ds < \infty$, implied by (4.51). Since $\|d\phi(s)\|_2 \leq \|d_C\phi(s)\|_2 + \|[C(s), \phi(s)]\|_2$ the integrability of the third term would follow from the integrability of $s^{1-a} \|[C(s), \phi(s)]\|_2^2$. But

$$\begin{aligned} \int_0^T s^{1-a} \|[C(s), \phi(s)]\|_2^2 ds &\leq c^2 \int_0^T s^{1-a} \|C(s)\|_6^2 \|\phi(s)\|_3^2 ds \\ &\leq c^2 \left(\sup_{0 \leq s \leq T} s^{1-a} \|C(s)\|_6^2 \right) \int_0^T \|\phi(s)\|_3^2 ds \\ &\leq (c\kappa_6)^2 |C|_T^2 \int_0^T \|\phi(s)\|_3^2 ds. \end{aligned} \quad (4.59)$$

Furthermore, using $0 = (a - (1/2)) - (a/2) + (1 - a)/2$ along with the interpolation $\|f\|_3^2 \leq \|f\|_2 \|f\|_6$ we find

$$\begin{aligned} \int_0^T \|\phi(s)\|_3^2 ds &\leq \int_0^T s^{a-(1/2)} \left(s^{-a/2} \|\phi(s)\|_2 \right) \left(s^{(1-a)/2} \|\phi(s)\|_6 \right) ds \\ &\leq T^{a-(1/2)} \left(\int_0^T s^{-a} \|\phi(s)\|_2^2 ds \right)^{1/2} \left(\int_0^T s^{1-a} \|\phi(s)\|_6^2 ds \right)^{1/2} \\ &\leq T^{a-(1/2)} \left(\int_0^T s^{-a} \|C(s)\|_{H_1}^2 ds \right)^{1/2} \left(\int_0^T s^{1-a} \|\phi(s)\|_6^2 ds \right)^{1/2}. \end{aligned} \quad (4.60)$$

The first integral is finite because $C(\cdot)$ has finite action. The second integral is finite by (4.52). The same argument shows that

$$\int_0^T \|B_C(s)\|_3^2 ds < \infty. \quad (4.61)$$

Combining (4.59) and (4.60) it follows that

$$\int_0^T s^{1-a} \|[C(s), \phi(s)]\|_2^2 ds < \infty. \quad (4.62)$$

This proves the integrability of the last term in (4.53) and completes the proof of Theorem 4.12. ■

4.5 Initial behavior, order 2

Theorem 4.13 (*Order 2*) Let $1/2 \leq a < 1$. Suppose that $C(\cdot)$ is a strong solution to the augmented equation (2.22) lying in \mathcal{P}_T^a and having finite strong a -action, i.e., (2.25) holds. Then

$$\begin{aligned} t^{2-a} \|C'(t)\|_2^2 + \int_0^t s^{2-a} e^{\beta_s^t} \left(\|d_C^* C'(s)\|_2^2 + \|d_C C'(s)\|_2^2 \right) ds & \quad (\text{Order 2}) \\ \leq (2-a)(1-a) e^{\beta_0^t} \int_0^t s^{-a} \left(\|B_C(s)\|_2^2 + \|\phi(s)\|_2^2 \right) ds. & \quad (4.63) \end{aligned}$$

The following L^6 bounds also hold.

$$\sup_{0 < t < T} t^{2-a} \left(\|B_C(t)\|_6^2 + \|\phi(t)\|_6^2 \right) < \infty. \quad (\text{Order 2}) \quad (4.64)$$

$$\int_0^T s^{2-a} \left(\|C'(s)\|_6^2 + \|d_C^* B_C(s)\|_6^2 + \|d_C \phi(s)\|_6^2 \right) ds < \infty. \quad (\text{Order 2}) \quad (4.65)$$

$$\int_0^T s^{2-a} \left(\|d\phi(s)\|_6^2 + \|[C(s), \phi(s)]\|_6^2 \right) ds < \infty. \quad (\text{Order 2}) \quad (4.66)$$

In preparation for the proof of the L^6 bounds we will need the following lemma.

Lemma 4.14 For a solution to the augmented equation (2.22) the following three (a -independent) inequalities hold.

$$\kappa^{-2} \|d_{C(s)}^* B_C(s)\|_6^2 \leq \|d_C d_C^* B_C\|_2^2 + \lambda(B_C) \|d_C^* B_C\|_2^2. \quad (4.67)$$

$$\kappa^{-2} \|d_{C(s)} \phi(s)\|_6^2 \leq \|d_C^* d_C \phi\|_2^2 + \|[B_C, \phi]\|_2^2 + \lambda(B_C) \|d_C \phi\|_2^2. \quad (4.68)$$

$$\begin{aligned} \kappa^{-2} \left(\|C'(s)\|_6^2 + \|d_{C(s)}^* B_C(s)\|_6^2 + \|d_{C(s)} \phi(s)\|_6^2 \right) & \quad (4.69) \\ \leq 3 \|d_C C'\|_2^2 + 2 \|d_C^* C'\|_2^2 + 3 \|[B_C, \phi]\|_2^2 + 2\lambda(B_C) \|C'\|_2^2. \end{aligned}$$

Proof. Use the GFS inequality (4.1) twice and the Bianchi identity twice, once for $d_C^* d_C^* B_C = 0$ and once in $d_C^2 \phi = [B_C, \phi]$, to find (4.67) and (4.68). In view of the identities

$$\begin{aligned} -d_C^* C'(s) &= d_C^* d_C \phi(s) & \text{and} \\ -d_C C'(s) &= d_C d_C^* B_C(s) + [B_C(s), \phi(s)], \end{aligned}$$

the first term on the right of each line in (4.67) and (4.68) can be expressed in terms of C' and $[B_C, \phi]$. We may add them and use (4.17) to find

$$\begin{aligned} & \kappa^{-2} \left(\|d_{C(s)}^* B_C(s)\|_6^2 + \|d_{C(s)} \phi(s)\|_6^2 \right) \\ & \leq \|d_C C' + [B_C, \phi]\|_2^2 + \|d_C^* C'\|_2^2 + \|[B_C, \phi]\|_2^2 + \lambda(B_C) \|C'\|_2^2 \\ & \leq 2\|d_C C'\|_2^2 + \|d_C^* C'\|_2^2 + 3\|[B_C, \phi]\|_2^2 + \lambda(B_C) \|C'\|_2^2. \end{aligned}$$

To this we may add the $\|C'(s)\|_6$ bound (4.22) to arrive at (4.69). ■

Proof of Theorem 4.13. From (4.45) in Theorem 4.11 we know that $\beta_0^T < \infty$. Let $\zeta(s) = \beta_0^s$. Since $\zeta'(s) = \beta(s)$, the inequality (4.13) implies that

$$\frac{d}{ds} \left(e^{-\zeta(s)} \|C'(s)\|_2^2 \right) + e^{-\zeta(s)} \left(\|d_C^* C'(s)\|_2^2 + \|d_C C'(s)\|_2^2 \right) \leq 0. \quad (4.70)$$

In Lemma 4.8 choose $b = a - 1$, $h(s) = 0$,

$$f(s) = e^{-\zeta(s)} \|C'(s)\|_2^2, \quad g(s) = e^{-\zeta(s)} \left(\|d_C^* C'(s)\|_2^2 + \|d_C C'(s)\|_2^2 \right)$$

and use (4.70) and (4.30) to find

$$\begin{aligned} & t^{2-a} e^{-\zeta(t)} \|C'(t)\|_2^2 + \int_0^t s^{2-a} e^{-\zeta(s)} \left(\|d_C^* C'(s)\|_2^2 + \|d_C C'(s)\|_2^2 \right) ds \\ & \leq (2-a) \int_0^t s^{1-a} e^{-\zeta(s)} \|C'(s)\|_2^2 ds. \end{aligned} \quad (4.71)$$

Since $\zeta(t) - \zeta(s) = \beta_s^t$, multiplication by $e^{\zeta(t)}$ gives

$$\begin{aligned} & t^{2-a} \|C'(t)\|_2^2 + \int_0^t s^{2-a} e^{\beta_s^t} \left(\|d_C^* C'(s)\|_2^2 + \|d_C C'(s)\|_2^2 \right) ds \\ & \leq (2-a) \int_0^t s^{1-a} e^{\beta_s^t} \|C'(s)\|_2^2 ds. \end{aligned} \quad (4.72)$$

From the coefficients in (4.19) and (4.27) one sees that $\beta(\sigma) - \alpha(\sigma) \geq 0$, and therefore $\beta_s^t - \alpha_s^t \leq \beta_0^t - \alpha_0^t$. Hence, using (4.51), one finds

$$\begin{aligned} (2-a) \int_0^t s^{1-a} e^{\beta_s^t} \|C'(s)\|_2^2 ds & \leq (2-a) e^{\beta_0^t - \alpha_0^t} \int_0^t s^{1-a} e^{\alpha_s^t} \|C'(s)\|_2^2 ds \\ & \leq (2-a)(1-a) e^{\beta_0^t - \alpha_0^t} \int_0^t s^{-a} e^{\alpha_s^t} \left(\|B_C(s)\|_2^2 + \|\phi(s)\|_2^2 \right) ds \\ & \leq (2-a)(1-a) e^{\beta_0^t} \int_0^t s^{-a} \left(\|B_C(s)\|_2^2 + \|\phi(s)\|_2^2 \right) ds. \end{aligned}$$

This proves (4.63).

To prove (4.64) multiply (4.58) by s to find

$$\begin{aligned} \kappa^{-2} s^{2-a} \left(\|B_C(s)\|_6^2 + \|\phi(s)\|_6^2 \right) \\ \leq s^{2-a} \|C'(s)\|_2^2 + s^{2-a} \|\phi(s)\|_2^2 + \{s\lambda(B_C(s))\} \{s^{1-a} \|B_C(s)\|_2^2\}. \end{aligned} \quad (4.73)$$

The first term on the right is bounded by (4.63). The second term is bounded by (4.36). The third term is bounded by (4.37) and (4.36) since $2 - 2a \leq 1$. This proves (4.64).

To prove (4.65) we can start with the a -independent inequality (4.69) and multiply it by s^{2-a} . The resulting first two terms on the right are integrable by (4.63). The third term is $s^{2-a} \| [B_C(s), \phi(s)] \|_2^2 \leq \left(c^2 s^{2-a} \|B_C(s)\|_6^2 \right) \|\phi(s)\|_3^2$, which is the product of a bounded function, by (4.64), times an integrable function, by (4.60). The last term is $\left(s\lambda(B_C(s)) \right) \left(s^{1-a} \|C'(s)\|^2 \right)$, which is a bounded function, by (4.37), times an integrable function, by (4.51). This proves (4.65).

Concerning the proof of (4.66), it suffices to prove the integrability of either one of the two terms because we already know that $\int_0^T s^{2-a} \|d\phi(s) + [C(s), \phi(s)]\|_6^2 ds < \infty$ by (4.65). We will prove the integrability of the second term and do this by invoking the GFS inequality (4.1). Thus, taking $\omega = [C(s), \phi(s)]$ in (4.1), it suffices to show that

$$\int_0^T s^{2-a} \left(\|d_C^*[C(s), \phi(s)]\|_2^2 + \|d_C[C(s), \phi(s)]\|_2^2 \right) ds < \infty \quad (4.74)$$

and

$$\int_0^T s^{2-a} \|B_C(s)\|_2^4 \| [C(s), \phi(s)] \|_2^2 ds < \infty. \quad (4.75)$$

To this end, observe first the identities

$$\begin{aligned} d_C^*[C, \phi] &= [C \lrcorner d_C \phi], \\ d_C[C, \phi] &= -[C \wedge d_C \phi] + [B_C + (1/2)[C \wedge C], \phi], \end{aligned}$$

which follow from $[d_C^* C, \phi] = [\phi, \phi] = 0$ and $d_C C = B_C + (1/2)[C \wedge C]$. Now

$\|[C \lrcorner d_C \phi]\|_2 \leq c\|C\|_6\|d_C \phi\|_3$ with the same bound for $\|C \wedge d_C \phi\|_2$. Therefore

$$\begin{aligned} & \int_0^T s^{2-a} \left(\|[C \lrcorner d_C \phi]\|_2^2 + \|C \wedge d_C \phi\|_2^2 \right) ds \\ & \leq 2c^2 \left(\sup_{0 \leq s \leq T} s^{1-a} \|C(s)\|_6^2 \right) \int_0^T s \|d_C \phi(s)\|_3^2 ds \\ & \leq (c\kappa_6)^2 |C|_T^2 \int_0^T s \|d_C \phi(s)\|_3^2 ds. \end{aligned} \quad (4.76)$$

Using now $1 = (a - (1/2)) + (1 - a)/2 + (2 - a)/2$ along with the interpolation $\|f\|_3^2 \leq \|f\|_2 \|f\|_6$ we find

$$\begin{aligned} \int_0^T s \|d_C \phi(s)\|_3^2 ds & \leq \int_0^T s^{a-(1/2)} \left(s^{(1-a)/2} \|d_C \phi(s)\|_2 \right) \left(s^{(2-a)/2} \|d_C \phi(s)\|_6 \right) ds \\ & \leq T^{a-(1/2)} \left(\int_0^T s^{1-a} \|d_C \phi(s)\|_2^2 ds \right)^{1/2} \left(\int_0^T s^{2-a} \|d_C \phi(s)\|_6^2 ds \right)^{1/2} \\ & < \infty \end{aligned} \quad (4.77)$$

by (4.53) and (4.65). Hence

$$\int_0^T s^{2-a} \left(\|[C \lrcorner d_C \phi]\|_2^2 + \|C \wedge d_C \phi\|_2^2 \right) ds < \infty.$$

Further,

$$\begin{aligned} & \int_0^T s^{2-a} \| [B_C + (1/2)[C \wedge C], \phi] \|_2^2 ds \\ & \leq \sup_{0 \leq s \leq T} \{ s \|B_C + (1/2)[C \wedge C]\|_3^2 \} \int_0^T s^{1-a} \|\phi(s)\|_6^2 ds. \end{aligned}$$

The integral in the last line is finite by (4.52). The supremum in that line is also finite as follows from the inequalities

$$s \|B_C(s)\|_3^2 \leq s^{a-(1/2)} \left(s^{\frac{1-a}{2}} \|B_C(s)\|_2 \right) \left(s^{\frac{2-a}{2}} \|B_C(s)\|_6 \right) \text{ and } \quad (4.78)$$

$$s \| [C(s) \wedge C(s)] \|_3^2 \leq c^2 s^{2a-1} \left(s^{\frac{1-a}{2}} \|C(s)\|_6 \right)^4 \leq c^2 s^{2a-1} (\kappa_6 |C|_T)^4, \quad (4.79)$$

since $|C|_T < \infty$ by the definition of \mathcal{P}_T^a while the two expressions in parentheses in line (4.78) are bounded on $(0, T]$ by respectively (4.36) and (4.64) for each $a \in [1/2, 1)$. This proves (4.74).

It remains only to prove (4.75). But

$$s^{2-a} \|B_C(s)\|_2^4 \| [C(s), \phi(s)] \|_2^2 = s^{2a-1} \left(s^{1-a} \|B_C(s)\|_2^2 \right)^2 \left(s^{1-a} \| [C(s), \phi(s)] \|_2^2 \right).$$

The first factor in parentheses is bounded by (4.36) while the last factor in parentheses is integrable by (4.62). This completes the proof of (4.66). ■

4.6 The case of infinite action

If a solution to the augmented variational equation (2.22) does not have finite a -action then our proofs of the estimates given in Theorems 4.12 and 4.13 do not hold. We are concerned in this section only with the case $a = 1/2$ because finite a -action always holds for $a > 1/2$. We are going to replace the first and second order initial behavior bounds of Theorems 4.12 and 4.13 by slightly weaker bounds. All of these are outgrowths of the inequality

$$\int_0^T s^{-1/2+\delta} \|C(s)\|_{H_1}^2 ds < \infty, \quad (4.80)$$

which holds for all paths $C(\cdot) \in \mathcal{P}_T^{1/2}$ and $\delta > 0$. This follows from (3.7) with $a = 1/2$, which shows that $\|C(s)\|_{H_1}^2 = o(s^{-1/2})$ as $s \downarrow 0$. The next theorem gives weaker information about the nature of the initial singularity than we obtained under the assumption of finite action, but is adequate for implementing a weaker version of the ZDS procedure: We will only be able to prove that $g(t) \in \mathcal{G}_{1,q}$ for $q < 3$ rather than in the smaller group $\mathcal{G}_{3/2}$.

Theorem 4.15 (*Order 1*) *Suppose that $C_0 \in H_{1/2}$ and that $C(\cdot)$ is a strong solution of (2.22) lying in $\mathcal{P}_T^{1/2}$. Define α_∞ and β_∞ as in (4.41). Then, for any $\delta > 0$, there holds*

$$\begin{aligned} & t^{(1/2)+\delta} \left(\|B_C(t)\|_2^2 + \|\phi(t)\|_2^2 \right) + \int_0^t s^{(1/2)+\delta} \|C'(s)\|_2^2 ds \\ & \leq (\alpha_\infty + (1/2) + \delta) \int_0^t s^{\delta-(1/2)} \left(\|B_C(s)\|_2^2 + \|\phi(s)\|_2^2 \right) ds \\ & < \infty. \end{aligned} \quad (\text{Order 1}) \quad (4.81)$$

Moreover

$$\int_0^T s^{\delta+(1/2)} \left(\|B_C(s)\|_6^2 + \|\phi(s)\|_6^2 \right) ds < \infty \quad \text{and} \quad (4.82)$$

$$\int_0^T s^{\delta+(1/2)} \left(\|d_C^* B_C(s)\|_2^2 + \|d_C \phi(s)\|_2^2 + \|d\phi(s)\|_2^2 \right) ds < \infty. \quad (4.83)$$

Proof. We start with the differential inequality (4.12). We cannot put the function $\alpha(\cdot)$ into an integrating factor, as we did under the assumption of finite action, because α is not integrable near zero. Instead we will apply the machinery of Lemma 4.8 directly to (4.12).

Let $d = \delta + (1/2)$. In (4.28) take $f(s) = \|B_C(s)\|_2^2 + \|\phi(s)\|_2^2$, $g(s) = \|C'(s)\|_2^2$ and $h(s) = \alpha(s)\|\phi(s)\|_2^2$. Then (4.12) asserts that (4.28) holds. Take $b = 1 - d$ in (4.30) to find

$$t^d f(t) + \int_0^t s^d \|C'(s)\|_2^2 ds \leq \int_0^t s^d \alpha(s) \|\phi(s)\|_2^2 ds + d \int_0^t s^{d-1} f(s) ds.$$

But $s^d \alpha(s) \leq s^{d-1} \alpha_\infty$ and $\|\phi(s)\|_2^2 \leq f(s)$. Hence the first inequality in (4.81) holds. By (4.39) one has $s^{\delta-(1/2)} \left(\|B_C(s)\|_2^2 + \|\phi(s)\|_2^2 \right) = O(s^{\delta-1})$ as $s \downarrow 0$, which is integrable. Since $\alpha_\infty < \infty$, by (4.41), the inequality (4.81) holds.

The proof of the weighted L^6 bound (4.82) imitates the proof of the corresponding bound for the finite action case: We may replace (4.58) (with $a = 1/2$) by

$$\begin{aligned} & \kappa^{-2} s^d \left(\|B_C(s)\|_6^2 + \|\phi(s)\|_6^2 \right) \\ & \leq s^d \|C'(s)\|_2^2 + s^d \|\phi(s)\|_2^2 + s^\delta \lambda(B_C(s)) \left(s^{1/2} \|B_C(s)\|_2^2 \right) \end{aligned} \quad (4.84)$$

with $d = 1/2 + \delta$. From (4.81) we see that $\int_0^T s^d \|C'(s)\|_2^2 ds < \infty$. The second term in (4.84) is bounded, by (4.39). Since

$$\int_0^T s^\delta \lambda(B_C(s)) ds = \int_0^T \left(s^\delta + O(s^{-1+\delta}) \right) ds < \infty, \quad (4.85)$$

the last term in (4.84) is integrable. This proves (4.82).

Concerning the inequality (4.83) observe that the integrability of the first two terms follows directly from (4.81) because $\|C'(s)\|_2^2 = \|d_C B_C(s)\|_2^2 +$

$\|d_C \phi(s)\|_2^2$. Just as in the proof of (4.53), it suffices to prove that $\int_0^T s^d \| [C(s), \phi(s)] \|_2^2 ds < \infty$. But

$$\begin{aligned} \int_0^T s^d \| [C(s), \phi(s)] \|_2^2 ds &\leq c^2 |C|_T^2 \int_0^T s^\delta \|\phi(s)\|_2 \|\phi(s)\|_6 ds \\ &\leq c^2 |C|_T^2 \left(\int_0^T s^{\delta-(1/2)} \|\phi(s)\|_2^2 ds \right)^{1/2} \left(\int_0^T s^{\delta+(1/2)} \|\phi(s)\|_6^2 ds \right)^{1/2} \\ &< \infty \end{aligned}$$

by (4.82). ■

Theorem 4.16 (*Order 2*) Suppose again that $C_0 \in H_{1/2}$ and that $C(\cdot)$ is a strong solution of (2.22) lying in $\mathcal{P}_T^{1/2}$. Define α_∞ and β_∞ as in (4.41). Then, for any $\delta > 0$, there holds

$$\begin{aligned} t^{(3/2)+\delta} \|C'(t)\|_2^2 + \int_0^t s^{(3/2)+\delta} \left(\|d_C^* C'(s)\|_2^2 + \|d_C C'(s)\|_2^2 \right) ds \\ \leq (\beta_\infty + d)(\alpha_\infty + (1/2) + \delta) \int_0^t s^{\delta-(1/2)} \left(\|B_C(s)\|_2^2 + \|\phi(s)\|_2^2 \right) ds \\ < \infty, \end{aligned} \quad (\text{Order 2}) \quad (4.86)$$

where $d = (3/2) + \delta$. Moreover

$$\sup_{0 < t < T} t^{(3/2)+\delta} \left(\|B_C(t)\|_6^2 + \|\phi(t)\|_6^2 \right) < \infty, \quad (4.87)$$

$$\int_0^T s^{(3/2)+\delta} \left(\|C'(s)\|_6^2 + \|d_C^* B_C(s)\|_6^2 + \|d_C \phi(s)\|_6^2 \right) ds < \infty \quad \text{and} \quad (4.88)$$

$$\int_0^T s^{(3/2)+\delta} \left(\|d\phi(s)\|_6^2 + \| [C(s), \phi(s)] \|_6^2 \right) ds < \infty. \quad (4.89)$$

Proof. The proof follows the pattern of that for the finite action case, Theorem 4.13, with modifications similar to those used in the proof of Theorem 4.15. We will omit the details. ■

4.7 High L^p bounds via Neumann domination

Our energy estimates were able to produce bounds on L^p norms of various functions on M for $2 \leq p \leq 6$. In this section we will derive L^p bounds of some functions for $6 \leq p \leq \infty$.

Theorem 4.17 ($p = \infty$) *Let $1/2 \leq a < 1$ and $0 < T < \infty$. Assume that either $M = \mathbb{R}^3$ or that M is convex in the sense of Definition 2.9. Suppose that $C(\cdot)$ is a strong solution to (2.22) lying in \mathcal{P}_T^a and having finite a -action. Then*

$$\int_0^T t^{(3/2)-a} \left(\|B_C(t)\|_\infty^2 + \|\phi(t)\|_\infty^2 \right) dt < \infty. \quad (4.90)$$

In particular,

$$\int_0^T t \left(\|B_C(t)\|_\infty^2 + \|\phi(t)\|_\infty^2 \right) dt < \infty \quad \text{if } a = 1/2 \quad (4.91)$$

and

$$\int_0^T \left(\|B_C(t)\|_\infty + \|\phi(t)\|_\infty \right) dt < \infty \quad \text{if } a > 1/2. \quad (4.92)$$

Furthermore,

$$\|B_C(t)\|_\infty = t^{-1+\frac{a-(1/2)}{2}} o(1) \quad \text{as } t \downarrow 0 \quad \text{if } 1/2 \leq a < 1. \quad (4.93)$$

In particular,

$$\|B_C(t)\|_\infty = o(t^{-1}) \quad \text{as } t \downarrow 0 \quad \text{if } a = 1/2. \quad (4.94)$$

Corollary 4.18 *If, in Theorem 4.17, we drop the hypothesis that $C(\cdot)$ has finite a -action then we still have*

$$\sup_{\epsilon \leq t \leq T} \|B_C(t)\|_\infty < \infty \quad (4.95)$$

for any $\epsilon > 0$.

Theorem 4.19 ($p < \infty$) *Under the same hypotheses as in Theorem 4.17, if $6 \leq p < \infty$ then*

$$\int_0^T t^{(3/2)-a-(3/p)} \left(\|B_C(t)\|_p^2 + \|\phi(t)\|_p^2 \right) dt < \infty. \quad (4.96)$$

In particular,

$$\int_0^T t^{1-(3/p)} \left(\|B_C(t)\|_p^2 + \|\phi(t)\|_p^2 \right) dt < \infty \quad \text{if } a = 1/2 \quad (4.97)$$

and

$$\int_0^T (\|B_C(t)\|_p + \|\phi(t)\|_p) dt < \infty \quad \text{for } 1/2 \leq a < 1. \quad (4.98)$$

Furthermore, for $1/2 \leq a < 1$, one has

$$\|B_C(t)\|_p + \|\phi(t)\|_p = t^{-1+(3/2p)+\frac{a-(1/2)}{2}} o(1) \quad \text{as } t \downarrow 0. \quad (4.99)$$

In particular,

$$\|B_C(t)\|_p = o(t^{-1+(3/2p)}) \quad \text{as } t \downarrow 0 \quad \text{if } a = 1/2. \quad (4.100)$$

Theorems 4.17 and 4.19 and Corollary 4.18 will be proven in the remainder of this subsection.

Remark 4.20 (Strategy) The method of proof of these theorems depends on showing that $|\phi(t, x)|$ and $|B_C(t, x)|$ satisfy partial differential inequalities of the form $\partial f / \partial t \leq \Delta_N f + \text{non-linear terms}$, where Δ_N is the Neumann Laplacian on real valued functions over M (or simply the Laplacian on real valued functions if $M = \mathbb{R}^3$). Unfortunately, $\phi(0, x)$ and $B_C(0, x)$ are, typically, distributions lying in $H_{-1/2}(M)$ and therefore use of their absolute values at time zero seems infeasible. Instead, we are going to represent $\phi(\sigma, x)$ as the solution to (4.6) over an interval $[s, t]$ with initial data $\phi(s, x)$ and with $s > 0$. We can then apply the Neumann domination techniques developed in [3] over the interval $[s, t]$ to derive s -dependent inequalities for $|\phi(t, x)|$, which can then be averaged with respect to s over the interval $(0, t)$.

The L^p bounds, for large p , that we will derive from these Neumann domination inequalities will rely on the energy estimates made in the previous subsections for low p .

4.7.1 Neumann domination with averaging.

Proposition 4.21 (*Neumann domination with averaging*) Assume that $M = \mathbb{R}^3$ or is the closure of a bounded open set in \mathbb{R}^3 with smooth boundary. Suppose that $A : (0, T] \rightarrow C^1(M; \Lambda^1 \otimes \mathfrak{k})$ is a time dependent 1-form on M which is continuous in the time variable. Let $\omega : (0, T) \rightarrow C^2(M; \Lambda^p \otimes \mathfrak{k})$ be a time dependent, \mathfrak{k} valued, p -form on M which is continuously differentiable in the time variable and satisfies the equation

$$\omega'(s, x) = \sum_{j=1}^N (\nabla_j^{A(s)})^2 \omega(s, x) + h(s, x), \quad 0 < s < T, \quad (4.101)$$

where $h \in C((0, T] \times M; \Lambda^p \otimes \mathfrak{k})$. If $M \neq \mathbb{R}^3$ then assume also that

$$\nabla_n |\omega(s, x)|^2 \leq 0 \quad \text{for } 0 < s < T, \quad x \in \partial M. \quad (4.102)$$

Then

$$\begin{aligned} |\omega(t, x)| &\leq t^{-1} \int_0^t e^{(t-s)\Delta_N} |\omega(s, \cdot)| ds(x) \\ &\quad + t^{-1} \int_0^t e^{(t-s)\Delta_N} s |h(s, \cdot)| ds(x). \end{aligned} \quad (4.103)$$

If $M = \mathbb{R}^3$ then the Neumann Laplacian in (4.103) should be replaced by the self-adjoint version Δ over \mathbb{R}^3 .

Proof. The proof of [3, Proposition 2.7] shows that under the hypotheses of this Proposition there holds

$$\begin{aligned} |\omega(t, x)| &\leq e^{(t-s)\Delta_N} |\omega(s, \cdot)| (x) \\ &\quad + \int_s^t e^{(t-\sigma)\Delta_N} |h(\sigma, \cdot)| d\sigma (x), \quad 0 < s < t < T. \end{aligned} \quad (4.104)$$

One need only take the origin in [3, Proposition 2.7] to be s in our present setting. The statement of [3, Proposition 2.7] includes the assumption that M is convex, which is used only to show that our hypothesis (4.102) holds in the cases of interest. We will prove separately, in Lemma 4.23, that (4.102) holds for our circumstances. The statement of [3, Proposition 2.7] further hypothesizes that M is compact. But when $M = \mathbb{R}^3$ the proof given there applies even more easily because one need not be concerned with boundary conditions. Instead one can allow $|\omega| + |\text{grad } \omega| \in L^2(\mathbb{R}^3)$ or even mild growth (e.g. polynomial) of these function as $x \rightarrow \infty$. These conditions will be satisfied for the functions $\omega = B$ or $\omega = \phi$ of interest to us.

The left side of (4.104) is independent of s . We may therefore average (4.104) over the interval $(0, t)$ to find

$$\begin{aligned} |\omega(t, x)| &\leq t^{-1} \int_0^t e^{(t-s)\Delta_N} |\omega(s, \cdot)| ds(x) \\ &\quad + t^{-1} \int_0^t \int_s^t e^{(t-\sigma)\Delta_N} |h(\sigma, \cdot)| d\sigma ds(x). \end{aligned} \quad (4.105)$$

Since $e^{(t-\sigma)\Delta_N}$ is a positivity preserving operator we can reverse the σ and s integrals in the last line to find $t^{-1} \int_0^t e^{(t-\sigma)\Delta_N} \sigma |h(\sigma, \cdot)| d\sigma$. This proves (4.103). ■

4.7.2 Pointwise bounds

The proofs of Theorems 4.17 and 4.19 depend on the following representation inequality.

Theorem 4.22 (*Pointwise bounds*) Assume that M is as in the statement of Theorem 4.17. Let $C(\cdot)$ be a smooth solution over $(0, T)$ to the augmented equation (2.22) satisfying either Neumann or Dirichlet boundary conditions, (2.23), resp. (2.24) in case $M \neq \mathbb{R}^3$. Then, for $0 < t < T$, the following pointwise bounds hold.

$$\begin{aligned} |B_C(t, x)| &\leq \frac{1}{t} \int_0^t e^{(t-s)\Delta_N} |B_C(s, \cdot)| ds (x) \\ &\quad + \frac{1}{t} \int_0^t e^{(t-s)\Delta_N} s \left| B_C(s) \# B_C(s) - [B_C(s), \phi(s)] \right| ds (x) \end{aligned} \quad (4.106)$$

and

$$\begin{aligned} |\phi(t, x)| &\leq \frac{1}{t} \int_0^t e^{(t-s)\Delta_N} |\phi(s, \cdot)| ds (x) \\ &\quad + \frac{1}{t} \int_0^t e^{(t-s)\Delta_N} s \left| [C(s) \lrcorner C'(s)] \right| ds (x). \end{aligned} \quad (4.107)$$

In case $M = \mathbb{R}^3$ the Neumann Laplacian Δ_N should be replaced by the self-adjoint Laplacian Δ here and in the following.

The proof of Theorem 4.22 depends on the following lemma.

Lemma 4.23 (*Normal derivatives*) Assume that M is as in the statement of Theorem 4.17 but $M \neq \mathbb{R}^3$. Let $C(\cdot)$ be a smooth solution to (2.22) over $(0, T)$ satisfying either Neumann boundary conditions (2.23) or Dirichlet boundary conditions (2.24). Then

$$\nabla_n |B_C(t)|^2 \leq 0, \quad 0 < t < T \quad \text{and} \quad (4.108)$$

$$\nabla_n |\phi(t)|^2 = 0, \quad 0 < t < T. \quad (4.109)$$

Proof. For fixed $t \in (0, T)$ let $\omega = B_C(t)$. In the case of Neumann boundary conditions (2.23) we have $\omega_{norm} = B_C(t)_{norm} = 0$ by (2.23) and $(d_C \omega)_{norm} = 0$ by the Bianchi identity. Therefore we may apply [3, Corollary 2.4] to find (4.108) in the case of Neumann boundary conditions.

In the case of Dirichlet boundary conditions we have $C(t)_{tan} = 0$ by (2.24) and therefore $dC(t)_{tan} = 0$ by [2, Equ. (3.19)]. Since also $(C(t) \wedge C(t))_{tan} = 0$ it follows that $B_C(t)_{tan} = 0$. In order to apply [3, Corollary 2.4] we need only show that $(d_C^* B_C(t))_{tan} = 0$. But the differential equation (2.22) shows that $(d_C^* B_C(t))_{tan} = -C'(t)_{tan} - (d_C \phi(t))_{tan}$. The first term is zero by differentiation of $C(t)_{tan} = 0$. The second term is zero by virtue of [2, Equ. (3.19)], since $\phi(t)_{tan} = 0$ by the assumption (2.24). Thus (4.108) holds for both Neumann and Dirichlet boundary conditions.

The proof of the identity (4.109) for the zero form $\phi(t)$ does not require convexity of M , unlike the proof of (4.108): At any boundary point x , the normal derivative of $|\phi(t, x)|^2$ is given by

$$\nabla_n |\phi(t, x)|^2 = \langle (d_C \phi(t, x))_{norm}, \phi(t, x) \rangle_{\mathfrak{t}} + \langle \phi(t, x), (d_C \phi(t, x))_{norm} \rangle_{\mathfrak{t}}.$$

This is zero in the case of Dirichlet boundary conditions since $\phi(t) = d^* C(t) = 0$ on ∂M by (2.24). In the case of Neumann boundary conditions we need to use the differential equation (2.22), which shows that $(d_C \phi(t))_{norm} = -C'(t)_{norm} - (d_C^* B_C(t))_{norm}$. The first term is zero by virtue of (2.23). The second term is zero by [2, Equ. (3.20)], since $B(t)_{norm} = 0$. ■

Proof of Theorem 4.22. Take $\omega(s) = B_C(s)$ in Proposition 4.21. The boundary condition (4.102) is satisfied in this case by Lemma 4.23. The identity (4.8) shows that (4.101) holds with $h(s) = B_C(s) \# B_C(s) - [B_C(s), \phi(s)]$. The role of the connection form A in Proposition 4.21 is played here by C . (4.106) now follows from (4.103).

For the proof of (4.107) take $\omega(t) = \phi(t)$ in Proposition 4.21. The required boundary condition (4.102) is satisfied, in accordance with Lemma 4.23. The identity (4.6) shows that (4.101) holds with $h(s) = -[C(s) \lrcorner C'(s)]$. (4.107) now follows from (4.103). ■

4.7.3 A convolution inequality and energy bounds

Lemma 4.24 (*A convolution inequality*) *Let $0 \leq c < 1$. Suppose that α and β are non-negative functions on $(0, T]$ such that*

$$\alpha(t) \leq (1/t) \int_0^t (t-s)^{-c} \beta(s) ds \quad \text{for } 0 < t \leq T. \quad (4.110)$$

Then for any real number $b < 2c$ there holds

$$\int_0^T t^b \alpha(t)^2 dt \leq \gamma \int_0^T s^{b-2c} \beta(s)^2 ds \quad (4.111)$$

for some constant γ depending only on b and c .

Proof. By the Schwarz inequality and (4.110) we have

$$\begin{aligned} t^b \alpha(t)^2 &\leq t^{b-2} \left(\int_0^t (t-s)^{-c} ds \right) \left(\int_0^t (t-s)^{-c} \beta(s)^2 ds \right) \\ &= \left(t^{b-2} t^{1-c} / (1-c) \right) \left(\int_0^t (t-s)^{-c} \beta(s)^2 ds \right). \end{aligned}$$

Therefore

$$\begin{aligned} (1-c) \int_0^T t^b \alpha(t)^2 dt &\leq \int_0^T t^{b-1-c} \left(\int_0^t (t-s)^{-c} \beta(s)^2 ds \right) dt \\ &= \int_0^T \left(\int_s^T t^{b-1-c} (t-s)^{-c} dt \right) \beta(s)^2 ds \\ &\leq \gamma \int_0^T s^{b-2c} \beta(s)^2 ds, \end{aligned}$$

wherein we have estimated the t integral by the change of variables $t = rs$ to find $\int_s^T t^{b-1-c} (t-s)^{-c} dt = s^{b-2c} \int_1^{T/s} r^{b-1-c} (r-1)^{-c} dr \leq s^{b-2c} \int_1^\infty r^{b-1-c} (r-1)^{-c} dr = s^{b-2c} \gamma$, which is finite because $b-2c-1 < -1$. ■

Lemma 4.25 (*Energy bounds*) Assume that $M = \mathbb{R}^3$ or is the closure of a bounded, convex, open set in \mathbb{R}^3 with smooth boundary. For $1/2 \leq a < 1$ there holds

$$\int_0^T s^{2-a} \left(\|B_C(s) \# B_C(s)\|_2^2 + \|[B_C(s), \phi(s)]\|_2^2 \right) ds < \infty. \quad (4.112)$$

$$s \left(\|B_C(s) \# B_C(s)\|_2 + \|[B_C(s), \phi(s)]\|_2 \right) = o(s^{a-(3/4)}) \quad \text{as } s \downarrow 0. \quad (4.113)$$

$$\int_0^T s^{2-a} \|[C(s) \lrcorner C'(s)]\|_2^2 ds < \infty. \quad (4.114)$$

$$s \|[C(s) \lrcorner C'(s)]\|_{3/2} = o(s^{a-(1/2)}) \quad \text{as } s \downarrow 0. \quad (4.115)$$

Proof. By Hölder we find $\|B(s)\#B(s)\|_2^2 \leq c^2\|B(s)\|_4^4 \leq c^2\|B(s)\|_2\|B(s)\|_6^3$. Hence

$$\begin{aligned} & \int_0^T s^{2-a}\|B(s)\#B(s)\|_2^2 ds \\ & \leq c^2 \int_0^T s^{a-(1/2)} \left(s^{(1-a)/2}\|B(s)\|_2 \right) \left(s^{(2-a)/2}\|B(s)\|_6 \right) \left(s^{1-a}\|B(s)\|_6^2 \right) ds. \end{aligned} \quad (4.116)$$

The first factor in parenthesis is bounded by (4.36). The second factor in parenthesis is bounded by virtue of the energy estimate (4.64). The third factor is integrable by (4.52). Hence the integral is finite. Notice that if $a = 1/2$ then there is no room to spare in these estimates, whereas if $a > 1/2$ there is an extra factor of s with strictly positive exponent.

Concerning the second term in (4.112) we have

$$\begin{aligned} \| [B(s), \phi(s)] \|_2^2 & \leq \|B(s)\|_4^2 \|\phi(s)\|_4^2 \\ & \leq \left(\|B(s)\|_2 \|\phi(s)\|_2 \right)^{1/2} \left(\|B(s)\|_6 \|\phi(s)\|_6 \right)^{1/2} \left(\|B(s)\|_6 \|\phi(s)\|_6 \right). \end{aligned}$$

Distribute the available factor s^{2-a} among the three factors, assigning $s^{(1-a)/2}$ to the first square root, $s^{(2-a)/2}$ to the second square root, and s^{1-a} to the last parenthesis, leaving a factor $s^{a-(1/2)}$ as before. Again we find two bounded products times an integrable product, as before. This completes the proof of (4.112).

The proof of (4.113) follows from the same kind of estimates. For the first term in (4.113), it suffices to show that $s^{3/2-2a}s^2\|B_C(s)\#B_C(s)\|_2^2 = o(1)$. But, as in the first line of this proof, we have

$$\begin{aligned} s^{3/2-2a}s^2\|B_C(s)\#B_C(s)\|_2^2 & \leq c^2 s^{3/2-2a}s^2\|B_C(s)\|_2\|B_C(s)\|_6^3 \\ & = c^2 \left(s^{(1-a)/2}\|B_C(s)\|_2 \right) \left(s^{(2-a)/2}\|B_C(s)\|_6 \right)^3 \end{aligned}$$

and all factors are bounded, while the first is $o(1)$. As in the proof of (4.112) a polarization-like argument applies to the second term in (4.113) also.

For a proof of (4.114) observe that

$$\begin{aligned} s^{2-a}\| [C(s)\lrcorner C'(s)] \|_2^2 & \leq c^2 s^{2-a}\|C(s)\|_6^2\|C'(s)\|_3^2 \\ & \leq c^2 s^{a-(1/2)} \left(s^{1-a}\|C(s)\|_6^2 \right) \left(s^{(1-a)/2}\|C'(s)\|_2 \right) \left(s^{(2-a)/2}\|C'(s)\|_6 \right). \end{aligned}$$

Over the interval $(0, T]$ the first factor is at most $T^{a-(1/2)}$, the second factor is bounded by (3.13) and the last two factors are square integrable by (4.51) and (4.65). This proves (4.114).

To prove (4.115) observe that (3.13) and (4.63) yield $s \| [C(s) \lrcorner C'(s)] \|_{3/2} \leq sc \| C(s) \|_6 \| C'(s) \|_2 \leq ss^{(a-1)/2} \kappa_6 |C|_t s^{(a-2)/2} = O(s^{a-(1/2)}) |C|_t$ for $0 < s \leq t$.

■

4.7.4 Proof of high L^p bounds

Proof of Theorem 4.17. Combine the two terms in (4.106) to find

$$|B_C(t, x)| \leq \frac{1}{t} \int_0^t e^{(t-s)\Delta_N} \beta(s) ds(x), \quad (4.117)$$

where

$$\beta(s, x) = |B_C(s, x)| + s \left(|B_C(s, x) \# B_C(s, x)| + |[B_C(s, x), \phi(s, x)]| \right). \quad (4.118)$$

Since $\|e^{(t-s)\Delta_N}\|_{2 \rightarrow \infty} \leq c_1(t-s)^{-3/4}$ we find

$$\|B_C(t)\|_\infty \leq \frac{c_1}{t} \int_0^t (t-s)^{-3/4} \|\beta(s)\|_2 ds. \quad (4.119)$$

Take $c = 3/4$ in Lemma 4.24 and take $b = 3/2 - a$. Then $b - 2c = -a < 0$. We can therefore apply Lemma 4.24. (4.111) yields

$$\begin{aligned} \int_0^T t^{(3/2)-a} \|B_C(t)\|_\infty^2 dt &\leq \gamma c_1^2 \int_0^T s^{-a} \|\beta(s)\|_2^2 ds, \\ &\leq c_2 \int_0^T \left\{ s^{-a} \|B_C(s)\|_2^2 + s^{2-a} \left(\|B_C(s) \# B_C(s)\|_2 + |[B_C(s), \phi(s)]| \right)^2 \right\} ds \end{aligned} \quad (4.120)$$

with $c_2 = 2\gamma c_1^2$. The integral of the first term is finite because C has finite a -action. The integral of the second term is finite by (4.112). This proves half of (4.90).

Starting with (4.107), the same argument as above shows that

$$\|\phi(t)\|_\infty \leq \frac{c_1}{t} \int_0^t (t-s)^{-3/4} \|\hat{\beta}(s)\|_2 ds, \quad (4.121)$$

where

$$\hat{\beta}(s, x) = |\phi(s, x)| + s | [C(s, x) \lrcorner C'(s, x)] |. \quad (4.122)$$

Lemma 4.24 applies with the same values of c and b and shows that

$$\int_0^T t^{(3/2)-a} \|\phi(t)\|_\infty^2 dt \leq \gamma c_1^2 \int_0^T s^{-a} \|\hat{\beta}(s)\|_2^2 ds. \quad (4.123)$$

The right hand side of (4.123) is finite since $\int_0^T s^{-a} \|\phi(s)\|_2^2 ds < \infty$ by finite a-action, while $\int_0^T s^{2-a} \| [C(s) \lrcorner C'(s)] \|_2^2 ds < \infty$ is proved in the energy estimate (4.114). This completes the proof of (4.90). Set $a = 1/2$ in (4.90) to derive (4.91).

For the proof of (4.92) let $f(t) = (\|B_C(t)\|_\infty + \|\phi(t)\|_\infty)$. Then, by the Schwarz inequality, $\left(\int_0^T f(t) dt \right)^2 \leq \int_0^T t^{a-(3/2)} dt \int_0^T t^{(3/2)-a} f(t)^2 dt < \infty$ because $a > 1/2$ and (4.90) holds. This proves (4.92).

The bound (4.93) is a pointwise (in t) bound rather than an integral bound and has a slightly different proof. Return to (4.119) and insert the following pointwise (in s) bounds on $\|\beta(s)\|_2$, which are a little different for the two types on terms in (4.118). We have $\|B_C(s)\|_2 = o(s^{(a-1)/2})$ by (4.36). On the other hand (4.113) shows that $s \left(\|B_C(s) \# B_C(s)\|_2 + \| [B_C(s), \phi(s)] \|_2 \right) = o(s^{a-(3/4)})$. Thus (4.119) shows that

$$\begin{aligned} \|B_C(t)\|_\infty &= t^{-1} \int_0^t (t-s)^{-3/4} \left(s^{(a-1)/2} + s^{a-(3/4)} \right) ds o(1) \\ &= \left(t^{-1+\frac{a-(1/2)}{2}} + t^{a-(3/2)} \right) o(1). \end{aligned}$$

Since $t^{a-(3/2)} = t^{-1} t^{\frac{a-(1/2)}{2}} O(1)$ the assertion (4.93) follows. Put $a = 1/2$ in (4.93) to find (4.94). ■

Proof of Corollary 4.18. Let $0 < \delta < T$. Over the interval $[\delta, T]$ the function $C(\cdot)$ is a strong solution lying in $\mathcal{P}_{[\delta, T]}^a$ (with obvious meaning for this notation). Since $\|C(t)\|_{H_1}$ is bounded on this interval we have $\int_\delta^T (s-\delta)^{-a} \|C(s)\|_{H_1}^2 ds < \infty$ for any $a < 1$. That is, $C(\cdot)$ has finite strong a-action over the interval $[\delta, T]$. We can apply Theorem 4.17 and conclude from (4.93) that $(t-\delta)\|B_C(t)\|_\infty$ is bounded over $(\delta, T]$. In particular, if we

choose $\delta = \epsilon/2$ and restrict t to $[\epsilon, T]$ we find that $(\epsilon/2)\|B_C(t)\|_\infty$ is bounded over this interval. This proves Corollary 4.18. ■

Proof of Theorem 4.19. In view of the heat kernel bound $\|e^{(t-s)\Delta_N}\|_{2 \rightarrow p} \leq c_1(t-s)^{-3/4+(3/2p)}$, the inequality (4.117) shows that

$$\|B_C(t)\|_p \leq \frac{c_1}{t} \int_0^t (t-s)^{-(3/4)+(3/2p)} \|\beta(s)\|_2 ds. \quad (4.124)$$

In Lemma 4.24 choose $c = 3/4 - (3/2p)$ and $b = 3/2 - a - (3/p)$. We have again $b - 2c = -a$, which is strictly negative. Lemma 4.24 now shows that

$$\int_0^T t^{(3/2)-a-(3/p)} \|B_C(t)\|_p^2 dt \leq \gamma c_1^2 \int_0^T s^{-a} \|\beta(s)\|_2^2 ds. \quad (4.125)$$

The right side is the same as that of (4.120), which we have already proven to be finite. This proves half of (4.96). Similarly, with these new values of c and b , the inequality (4.121) changes to

$$\|\phi(t)\|_p \leq \frac{c_1}{t} \int_0^t (t-s)^{-(3/4)+(3/2p)} \|\hat{\beta}(s)\|_2 ds, \quad (4.126)$$

and therefore, by Lemma 4.24,

$$\int_0^T t^{(3/2)-a-(3/p)} \|\phi(t)\|_p^2 dt \leq \gamma c_1^2 \int_0^T s^{-a} \|\hat{\beta}(s)\|_2^2 ds. \quad (4.127)$$

The right side has already been shown to be finite in the discussion after (4.123). This completes the proof of (4.96).

Put $a = 1/2$ in (4.96) to find (4.97). Since $(3/2) - a - (3/p) < 1$ for all $a \in [1/2, 1)$ the inequality (4.98) follows from the Schwarz inequality and (4.96) just as in the proof of (4.92).

For the proof of the pointwise bounds (4.99) a slight deviation from these choices of c will be needed. We can choose again $c = (3/4) - (3/2p)$ to find

$$\begin{aligned} \|B_C(t)\|_p &= t^{-1} \int_0^t (t-s)^{-(3/4)+(3/p)} \left(s^{(a-1)/2} + s^{a-(3/4)} \right) ds \, o(1) \\ &= \left(t^{-1+(3/p)+\frac{a-(1/2)}{2}} + t^{a-(3/2)+(3/p)} \right) o(1) \\ &= \left(t^{-1+(3/p)+\frac{a-(1/2)}{2}} \right) o(1) \quad \text{as } t \downarrow 0. \end{aligned}$$

Here we have used again $t^{a-(3/2)} = O(t^{-1+\frac{a-(1/2)}{2}})$. This proves half of (4.99)

For the corresponding bound on $\|\phi(t)\|_p$ we must go back to the Neumann pointwise bound (4.107), which we may write as

$$|\phi(t, x)| \leq \frac{1}{t} \int_0^t e^{(t-s)\Delta_N} \hat{\beta}(s) ds (x), \quad (4.128)$$

with $\hat{\beta}$ as given in (4.122). For the first term in $\hat{\beta}$ we have $\|\phi(s)\|_2 = o(s^{(a-1)/2})$ because $C \in \mathcal{P}_T^a$. Therefore, choosing $c = (3/4) - (3/2p)$ again, we find

$$\begin{aligned} \left\| \frac{1}{t} \int_0^t e^{(t-s)\Delta_N} |\phi(s)| ds (\cdot) \right\|_p &\leq \frac{c_1}{t} \int_0^t (t-s)^{-(3/4)+(3/p)} \|\phi(s)\|_2 ds \\ &= \frac{1}{t} \int_0^t (t-s)^{-(3/4)+(3/p)} s^{(a-1)/2} ds o(1) \\ &= t^{-1+(3/p)+\frac{a-(1/2)}{2}} o(1). \end{aligned}$$

The second term in $\hat{\beta}$ must be estimated differently because we have only the $L^{3/2}$ bound (4.115). We must use the heat operator bound $\|e^{(t-s)\Delta}\|_{3/2 \rightarrow p} \leq c_1(t-s)^{-1+(3/p)}$. (We were not able to use this in case $p = \infty$ because the kernel $(t-s)^{-1}$ is not integrable.) Thus, in view of (4.115), we have

$$\begin{aligned} &\left\| \frac{1}{t} \int_0^t e^{(t-s)\Delta_N} |s[C(s) \lrcorner C'(s)]| ds (\cdot) \right\|_p \\ &\leq \frac{c_1}{t} \int_0^t (t-s)^{-1+(3/p)} \|s[C(s) \lrcorner C'(s)]\|_{3/2} ds \\ &\leq \frac{c_1}{t} \int_0^t (t-s)^{-1+(3/p)} s^{a-(1/2)} ds o(1) \\ &= t^{a-(3/2)+(3/p)} o(1), \end{aligned}$$

which is also $t^{-1+(3/p)+\frac{a-(1/2)}{2}} o(1)$. This completes the proof of (4.99) and of Theorem 4.19. ■

Remark 4.26 (Missing $\|\phi(t)\|_\infty$) Among the initial behaviors that have been described in Theorems 4.17 and 4.19 the behavior $\|\phi(t)\|_\infty = o(t^{-1+\frac{a-(1/2)}{2}})$ is noticeably missing. It is the ϕ analog of (4.93). Our proof for ϕ is not symmetrical to our proof for $B_C(t)$ because the energy bound (4.115), with

$3/2$ replaced by some $p > 3/2$, would require third order energy estimates for C , which are not in this paper. We are forced thereby to use the index $3/2$ in (4.115). But the heat operator bound $\|e^{t\Delta_N}\|_{3/2 \rightarrow \infty} = O(t^{-1})$ is not integrable and therefore cannot be used in the argument that produced (4.93). It is very likely that third order energy estimates would succeed in proving this $\|\phi(t)\|_\infty$ bound. But it is not needed in this paper.

5 Gauge groups

5.1 Notation and statements

Notation 5.1 (Gauge Groups) In this section we will take M to be either all of \mathbb{R}^3 or the closure of a bounded open set in \mathbb{R}^3 with smooth boundary. We will not require M to be convex. Denote by Δ the self adjoint version of the Laplacian on \mathfrak{k} valued 1-forms on \mathbb{R}^3 in case $M = \mathbb{R}^3$, or the Dirichlet or Neumann Laplacian on $L^2(M; \Lambda^1 \otimes \mathfrak{k})$ in case $M \neq \mathbb{R}^3$. The Dirichlet and Neumann domains were defined in Definition 2.3. See [2] for further discussion of these domains. For a measurable function $g : M \rightarrow K \subset \text{End } \mathcal{V}$ the weak derivatives $\partial_j g(x)$ are well defined, $\text{End } \mathcal{V}$ valued distributions on M^{int} (or on \mathbb{R}^3 if $M = \mathbb{R}^3$). We will say that $g \in W_1(M; K)$ if $\|g - I_{\mathcal{V}}\|_2 < \infty$ and the derivatives $\partial_j g \in L^2(M; \text{End } \mathcal{V})$. If $g \in W_1$ and $M \neq \mathbb{R}^3$ then the restriction $g|_{\partial M}$ is well defined a.e. with respect to surface measure by a Sobolev trace theorem.

Write $g^{-1}dg$ for the 1-form $\sum_{j=1}^3 \left(g(x)^{-1} \partial_j g(x) \right) dx^j$. The coefficients $g(x)^{-1} \partial_j g(x)$ lie in $\mathfrak{k} \subset \text{End } \mathcal{V}$ for a.e. $x \in M$. Thus $g^{-1}dg$ is an a.e. defined \mathfrak{k} valued 1-form on M . Let $D = (1 - \Delta)^{1/2}$. We may apply powers of the operator D to the \mathfrak{k} valued 1-form $g^{-1}dg$ and will write $g^{-1}dg \in H_a$ if $g^{-1}dg \in \mathcal{D}(D^a)$. Define

$$\|g^{-1}dg\|_{H_a} = \|D^a(g^{-1}dg)\|_2, \quad 0 \leq a \leq 1. \quad (5.1)$$

This norm has already been defined for general \mathfrak{k} valued 1-forms in Definition 2.3. The Sobolev space $H_a = H_a(M; \Lambda^1 \otimes \mathfrak{k})$ encodes Neumann or Dirichlet boundary conditions in accordance with Definition 2.3 when $M \neq \mathbb{R}^3$ and $1/2 \leq a \leq 1$.

In addition to the sets of gauge functions $g \in W_1(M; K)$ for which $\|g^{-1}dg\|_{H_a} < \infty$ we will need to use sets of gauge functions $g \in W_1(M; K)$ for which $\|g^{-1}dg\|_{L^p(M; \Lambda^1 \otimes \mathfrak{k})} < \infty$. Our proofs will make important use of these

as preliminary target spaces for the gauge functions arising in the ZDS procedure. We want to consider the following six kinds of sets of gauge functions. Some of these sets will be shown to be groups under pointwise multiplication.

For $0 \leq a \leq 1$ we let

$$\mathcal{G}_{1+a}(\mathbb{R}^3) = \left\{ g \in W_1(\mathbb{R}^3; K) : g^{-1}dg \in H_a(\mathbb{R}^3; \Lambda^1 \otimes \mathfrak{k}) \right\} \quad (5.2)$$

and, if $M \neq \mathbb{R}^3$

$$\mathcal{G}_{1+a}^N = \left\{ g \in W_1(M; K) : g^{-1}dg \in H_a(M; \Lambda^1 \otimes \mathfrak{k}) \right\}, \quad (5.3)$$

$$\mathcal{G}_{1+a}^D = \left\{ g \in W_1(M; K) : g^{-1}dg \in H_a(M; \Lambda^1 \otimes \mathfrak{k}), \ g = I_V \text{ on } \partial M \right\}. \quad (5.4)$$

For $2 \leq p \leq \infty$ we let

$$\mathcal{G}_{1,p}(\mathbb{R}^3) = \left\{ g \in W_1(\mathbb{R}^3; K) : g^{-1}dg \in L^p(\mathbb{R}^3; \Lambda^1 \otimes \mathfrak{k}) \right\}, \quad (5.5)$$

and, if $M \neq \mathbb{R}^3$

$$\mathcal{G}_{1,p}^N = \left\{ g \in W_1(M; K) : g^{-1}dg \in L^p(M; \Lambda^1 \otimes \mathfrak{k}) \right\}, \quad (5.6)$$

$$\mathcal{G}_{1,p}^D = \left\{ g \in W_1(M; K) : g^{-1}dg \in L^p(M; \Lambda^1 \otimes \mathfrak{k}), \ g = I_V \text{ on } \partial M \right\}. \quad (5.7)$$

In case $p = \infty$ we require also that $g^{-1}dg$ be continuous. Henceforth \mathcal{G}_{1+a} will refer to any of the three sets (5.2) - (5.4) and $\mathcal{G}_{1,p}$ will refer to any of the three sets (5.5) - (5.7). For functions g and h in $W_1(M; K)$ define

$$\rho_a(g, h) = \|g^{-1}dg - h^{-1}dh\|_{H_a} + \|g - h\|_2, \quad 0 \leq a \leq 1 \quad (5.8)$$

and

$$\rho_p(g, h) = \|g^{-1}dg - h^{-1}dh\|_p + \|g - h\|_2, \quad 2 \leq p \leq \infty. \quad (5.9)$$

ρ_a and ρ_p are clearly metrics on the sets \mathcal{G}_{1+a} and $\mathcal{G}_{1,p}$ respectively.

We will prove that the sets $\mathcal{G}_{1,p}$ and \mathcal{G}_{1+a} are complete topological groups under pointwise multiplication in their respective metrics, ρ_p or ρ_a , for $2 \leq p \leq \infty$ and $1/2 \leq a \leq 1$. We will ignore the case $p = \infty$ in all of the following statements because the proofs in this case are elementary.

Theorem 5.2 *$\mathcal{G}_{1,p}$ is a complete topological group under pointwise multiplication in the metric ρ_p , when $2 \leq p < \infty$.*

Theorem 5.3 \mathcal{G}_{1+a} is a complete topological group under pointwise multiplication in the metric ρ_a , when $1/2 \leq a \leq 1$.

Theorem 5.4 Let $0 \leq b \leq 1$ and let $3 \leq p \leq \infty$. If $g \in \mathcal{G}_{1,p}$ then the adjoint action

$$u \mapsto (Ad\ g)u = gug^{-1}, \quad u \in H_b \quad (5.10)$$

is a bounded operator on H_b . The representation

$$\mathcal{G}_{1,p} \ni g \mapsto (Ad\ g : H_b \rightarrow H_b) \quad (5.11)$$

is strongly continuous if $p = 3$. It is norm continuous if $p > 3$ and M has finite volume.

Corollary 5.5 Let $0 \leq b \leq 1$. The representation

$$\mathcal{G}_{1+a} \ni g \mapsto (Ad\ g : H_b \rightarrow H_b) \quad (5.12)$$

is strongly continuous if $a = 1/2$ and norm continuous if $1/2 < a \leq 1$.

The proofs will be given in the next four subsections.

Remark 5.6 The changeover from norm continuity to strong continuity in Corollary 5.5 as $a \downarrow 1/2$ is typical of the contrasts that we have seen before between $a > 1/2$ and $a = 1/2$. By Sobolev, $\mathcal{G}_{1+a} \subset \mathcal{G}_{1,p}$ if $1/p = 1/2 - a/3$. Thus $p = 3$ corresponds to $a = 1/2$ in the sense of these containments. Theorem 5.4 also shows this loss of norm continuity as $p \downarrow 3$.

Remark 5.7 (More about boundary conditions for \mathcal{G}_{1+a}) If $g^{-1}dg \in H_a$ and $a > 1/2$ then $g^{-1}dg|_{\partial M}$ is well defined almost everywhere on ∂M by well known Sobolev restriction theorems. In this case the boundary condition $(g^{-1}dg)_{tan} = 0$ in the Dirichlet case (5.4) is consistent with the condition $g|_M = I_V$ in the definition (5.4). In the Neumann case (5.3) one has $(g^{-1}dg)_{norm} = 0$ if $a > 1/2$ and this is the only boundary condition forced on elements of \mathcal{G}_{1+a}^N by (5.3) when $a > 1/2$.

But in the critical case, $a = 1/2$, the restriction $g^{-1}dg|_{\partial M}$ is ill defined. Nevertheless the boundary conditions $(g^{-1}dg)_{tan} = 0$, resp. $(g^{-1}dg)_{norm} = 0$ hold in a mean sense by Fujiwara's theorem [10]. Thus an element in the space $\mathcal{G}_{3/2}$ (Neumann) satisfies Neumann boundary conditions in a mean sense because of the requirement that $g^{-1}dg \in H_{1/2}$ (Neumann), while an element

of $\mathcal{G}_{3/2}$ (Dirichlet) satisfies both $g|\partial M = I_V$ pointwise almost everywhere, and also $(g^{-1}dg)_{tan} = 0$ in a mean sense. These functional analytic meanings of the boundary conditions will not be needed in this paper, but will be needed in [17] (Localization), where they will be discussed further. In case $a < 1/2$ the spaces H_a do not force any boundary conditions on $g^{-1}dg$.

Remark 5.8 (Boundary conditions for $\mathcal{G}_{1,p}$) The L^p norm imposes no boundary conditions on $g^{-1}dg$. Thus if $g \in \mathcal{G}_{1,p}^D(M)$ then the definition (5.7) imposes only the boundary condition $g = I_V$ on ∂M , while if $g \in \mathcal{G}_{1,p}^N$ then no condition need be satisfied at the boundary by g or dg .

5.2 Multiplier bounds for $Ad g$

The proof of Theorem 5.2 requires little more than use of Hölder inequalities. But the proof of Theorem 5.3 requires use of multiplier bounds on Sobolev spaces. In three dimensions the Sobolev $H_{3/2}$ norm of a function just fails to control its supremum norm, with the result that multiplication by such a function is not a bounded operator on Sobolev spaces. However we are interested in multiplication by the $End \mathcal{V}$ valued function $Ad g(x)$, which is a bounded function because $g(x)$ lies in the compact group K . Consequently we are able to derive better multiplier bounds for these functions than one would expect in the critical case.

Proposition 5.9 (*Multiplier bounds for $Ad g$*) Suppose that $g \in H_1(M; K)$ and that $g^{-1}dg \in L^3(M)$. Let $b \in [0, 1]$. Then, for any form $u \in L^2(M; \Lambda^1 \otimes \mathfrak{k})$, there holds

$$\|(Ad g)u\|_{H_b} \leq \left(1 + c_1 \|g^{-1}dg\|_3\right) \|u\|_{H_b} \quad \text{and} \quad (5.13)$$

$$\|(Ad g - 1)u\|_{H_b} \leq \left(\|Ad g - 1\|_\infty + c_1 \|g^{-1}dg\|_3\right) \|u\|_{H_b} \quad (5.14)$$

for a constant c_1 depending only on the commutator bound c and a Sobolev constant. Let $0 < \delta_1 < 3/2$. Define $p_1 = 3/\delta_1$. Then

$$\|(Ad g - 1)u\|_{H_b} \leq \left(\kappa_{\delta_1} \|Ad g - 1\|_{p_1} + c_1 \|g^{-1}dg\|_3\right) \|u\|_{H_{b+\delta_1}} \quad (5.15)$$

for some Sobolev constant κ_{δ_1} .

The proof depends on the following standard interpolation lemma.

Lemma 5.10 (*Complex interpolation*) Suppose that S is a bounded operator on a complex Hilbert space H . Let D be a self-adjoint operator on H such that $D \geq 1$ and DSD^{-1} is bounded. Then, for $0 \leq b \leq 1$, the operator D^bSD^{-b} is bounded by $\max(\|S\|, \|DSD^{-1}\|)$.

Proof. Let $u \in H$ and let v be in the spectral subspace of D for the interval $[1, \lambda]$ with $\lambda < \infty$. Then, for $z = x + iy$ in the strip $0 \leq x \leq 1$, the function $f(z) \equiv (SD^{-z}u, D^{\bar{z}}v)$ is bounded and continuous and analytic in the interior. On the left hand edge of the strip we have $|f(0 + iy)| \leq \|S\|\|u\|\|v\|$ because D^{iy} is unitary. On the right hand edge of the strip we have

$$|f(1 + iy)| = |(SD^{-1}D^{-iy}u, DD^{-iy}v)| \leq \|DSD^{-1}\|\|u\|\|v\|.$$

By the three lines theorem $f(b + iy)$ is bounded by the maximum of the right sides of the last two inequalities. In particular, at $y = 0$, we have $|(SD^{-b}u, D^b v)| \leq \gamma\|u\|\|v\|$, where γ is the maximum of $\|S\|$ and $\|DSD^{-1}\|$. Since this holds for all $u \in H$ and for all v in a core for D , it follows that $\|D^bSD^{-b}\| \leq \gamma$. ■

Lemma 5.11 Let $D = (1 - \Delta)^{1/2}$ as in Section 5.1 and let $u \in L^2(M; \Lambda^1 \otimes \mathfrak{k})$. If $R = Ad g$ or $R = Ad g - 1$ and $\delta \geq 0$ then

$$\|RD^{-\delta}u\|_{H_1} \leq (m + c_1\|g^{-1}dg\|_3)\|u\|_{H_1}, \quad (5.16)$$

where $c_1 = 2^{1/2}c\kappa_6$ and

$$m = \max\{\|RD^{-\delta}|L^2(M; \Lambda^j \otimes \mathfrak{k})\| : j = 0, 1, 2\}. \quad (5.17)$$

Proof. We will need to carry out the computations in terms of d and d^* rather than in terms of the partial derivatives ∂_j because only the former respect the boundary conditions adequately. Let $h = g^{-1}dg$. Observe first the identities

$$d\{(Ad g)u\} = Ad g\left(du + [h \wedge u]\right) \quad \text{and} \quad (5.18)$$

$$d^*\{(Ad g)u\} = Ad g\left(d^*u + [h \lrcorner u]\right), \quad (5.19)$$

which may be derived as follows: For an element $\alpha \in \mathfrak{k}$ we have, at each point $x \in M^{int}$,

$$\begin{aligned} \partial_j\{g(x)\alpha g(x)^{-1}\} &= g(x)[g(x)^{-1}\partial_j g(x), \alpha]g(x)^{-1} \\ &= (Ad g(x))(ad h_j(x))\alpha, \end{aligned}$$

where $h_j(x) = g(x)^{-1}\partial_j g(x)$. Hence

$$d\{(Ad g)u\} = (Ad g)du + \sum_{j=1}^3 dx^j \wedge (Ad g)(ad h_j)u,$$

which is (5.18). The derivation of (5.19) is similar, using $-d^*\{(Ad g)u\} = \sum_{j=1}^3 \partial_j \{(Ad g)u_j\}$, where $u = \sum_{j=1}^3 u_j dx^j$.

Thus if either $R = (Ad g)$ or $R = Ad g - 1$ we have

$$d(Ru) = Rdu + (Ad g)[h \wedge u] \quad (5.20)$$

$$d^*(Ru) = Rd^*u + (Ad g)[h \lrcorner u] \quad (5.21)$$

for elements $u \in L^2(M; \Lambda^1 \otimes \mathfrak{k})$ which are in the domain of d and of d^* . Therefore

$$\begin{aligned} \|DRu\|_2^2 &= \|dRu\|_2^2 + \|d^*Ru\|_2^2 + \|Ru\|_2^2 \\ &= \|Rdu + (Ad g)[h \wedge u]\|_2^2 + \|Rd^*u + (Ad g)[h \lrcorner u]\|_2^2 + \|Ru\|_2^2 \\ &\leq \left(\|Rdu\|_2 + \|[h \wedge u]\|_2\right)^2 + \left(\|Rd^*u\|_2 + \|[h \lrcorner u]\|_2\right)^2 + \|Ru\|_2^2 \\ &= \left(\|Rdu\|_2^2 + \|Rd^*u\|_2^2 + \|Ru\|_2^2\right) \\ &\quad + 2\|Rdu\|_2\|[h \wedge u]\|_2 + 2\|Rd^*u\|_2\|[h \lrcorner u]\|_2 + \|[h \wedge u]\|_2^2 + \|[h \lrcorner u]\|_2^2 \\ &\leq \left(\|Rdu\|_2^2 + \|Rd^*u\|_2^2 + \|Ru\|_2^2\right) + 2\left(\|Rdu\|_2^2 + \|Rd^*u\|_2^2\right)^{1/2} \mu + \mu^2, \end{aligned}$$

where $\mu^2 = \|[h \wedge u]\|_2^2 + \|[h \lrcorner u]\|_2^2$. Now $\mu^2 \leq 2(c\|h\|_3\|u\|_6)^2 \leq 2(c\kappa_6\|h\|_3\|Du\|_2)^2$. So $\mu \leq \nu\|Du\|_2$ where $\nu = 2^{1/2}c\kappa_6\|h\|_3 = c_1\|h\|_3$. Hence

$$\begin{aligned} \|DRu\|_2^2 &\leq \left(\|Rdu\|_2^2 + \|Rd^*u\|_2^2 + \|Ru\|_2^2\right) \\ &\quad + 2\left(\|Rdu\|_2^2 + \|Rd^*u\|_2^2\right)^{1/2} \nu\|Du\|_2 + \nu^2\|Du\|_2^2. \end{aligned} \quad (5.22)$$

Now replace u by $D^{-\delta}u$ and observe that d and d^* commute with D^2 and therefore also with $D^{-\delta}$. Thus, with m defined by (5.17), we have $\|RdD^{-\delta}u\|_2 = \|RD^{-\delta}du\|_2 \leq m\|du\|_2$ with similar inequalities when d is replaced by d^* or by I . One should note that D is acting in these inequalities on 0, 1 or 2-forms. Then we find

$$\|DRD^{-\delta}u\|_2^2 \leq m^2\|Du\|_2^2 + 2m\|Du\|_2\nu\|D^{-\delta}Du\|_2 + \nu^2\|D^{-\delta}Du\|_2^2. \quad (5.23)$$

Since $D \geq I$ on 1-forms we can drop the factor $D^{-\delta}$ in the last two terms of (5.23) and then take the square root to find $\|DRD^{-\delta}u\|_2 \leq (m + \nu)\|Du\|_2$, which is exactly (5.16). ■

Proof of Proposition 5.9. Choose H in Lemma 5.10 to be the complexification of $L^2(M; \Lambda^1 \otimes \mathfrak{k})$. To prove (5.13) and (5.14) take $\delta = 0$ in (5.16). First choose $R = Ad g$ and $S = R$ in Lemma 5.10. Then $m \equiv \|R\|_{2 \rightarrow 2} = \|Ad g\|_{2 \rightarrow 2} = 1$. Thus

$$\max(\|R\|_{2 \rightarrow 2}, \|DRD^{-1}\|_{2 \rightarrow 2}) = 1 + c_1 \|g^{-1}dg\|_3. \quad (5.24)$$

(5.13) now follows from Lemma 5.10.

Now choose $R = Ad g - 1$ and again $S = R$. Since $m \equiv \|R\|_{2 \rightarrow 2} = \|Ad g - 1\|_\infty$, the inequality (5.14) now follows the same way.

For the proof of (5.15) take $\delta = \delta_1$ and $R = Ad g - 1$ in (5.16). We apply Lemma 5.10 this time to the operator $S = RD^{-\delta_1}$. In view of (5.16), we need only verify that $\|RD^{-\delta_1}\|_{2 \rightarrow 2} \leq \kappa_{\delta_1} \|Ad g - 1\|_{p_1}$. For this, observe that $q^{-1} + p_1^{-1} = 1/2$ implies that $q^{-1} = (1/2) - (\delta_1/3)$ and therefore, by Hölder and Sobolev,

$$\begin{aligned} \|RD^{-\delta_1}v\|_2 &= \|(Ad g - 1)D^{-\delta_1}v\|_2 \\ &\leq \|Ad g - 1\|_{p_1} \|D^{-\delta_1}v\|_q \\ &\leq \|Ad g - 1\|_{p_1} \kappa_{\delta_1} \|v\|_2 \end{aligned} \quad (5.25)$$

for all $v \in L^2(M; \Lambda^j \otimes \mathfrak{k})$, $j = 0, 1, 2$. This completes the proof of Proposition 5.9. ■

5.3 $\mathcal{G}_{1,p}$ and \mathcal{G}_{1+a} are groups

Lemma 5.12

a) For $2 \leq p \leq \infty$, $\mathcal{G}_{1,p}$ is a group under pointwise multiplication. ρ_p is right invariant and right translation is continuous.

b) For $1/2 \leq a \leq 1$, \mathcal{G}_{1+a} is a group under pointwise multiplication. Right translation is continuous.

The proof depends on the following properties of the metrics ρ_p and ρ_a .

Lemma 5.13 (*Properties of ρ_p and ρ_a .*) Suppose that $2 \leq p \leq \infty$. If g and h are in $\mathcal{G}_{1,p}$ then

$$\rho_p(gh, e) \leq \rho_p(g, e) + \rho_p(h, e) \quad (5.26)$$

$$\rho_p(g^{-1}, e) = \rho_p(g, e) \quad (5.27)$$

$$\rho_p(gk, hk) = \rho_p(g, h) \quad \forall k \in \mathcal{G}_{1,p} \quad (5.28)$$

$$\rho_p(hkh^{-1}, e) \leq \rho_p(k, e) + \|(Ad k - 1)h^{-1}dh\|_p. \quad (5.29)$$

Suppose that $1/2 \leq a \leq 1$. If g and h are in \mathcal{G}_{1+a} then

$$\rho_a(gh, e) \leq \rho_a(g, e) + \rho_a(h, e) + c_2 \rho_a(g, e) \rho_a(h, e) \quad (5.30)$$

$$\rho_a(g^{-1}, e) \leq \rho_a(g, e) + c_2 \rho_a(g, e)^2 \quad (5.31)$$

$$\rho_a(gk, hk) \leq (1 + c_2 \rho_a(k, e)) \rho_a(g, h) \quad \forall k \in \mathcal{G}_{1+a} \quad (5.32)$$

$$\rho_a(hgh^{-1}, e) \leq \left(1 + c_1 \rho_3(h, e)\right) \left(\rho_a(g, e) + \|(Ad g^{-1} - 1)(h^{-1}dh)\|_{H_a}\right) \quad (5.33)$$

with constants c_1 and c_2 depending only on Sobolev constants and the commutator bound c . Note: (5.33) holds for all $a \in [0, 1]$ in this form.

Proof. The proof of each assertion relies on one of the following identities.

$$(hg)^{-1}d(hg) = g^{-1}dg + (Ad g^{-1})(h^{-1}dh) \quad (5.34)$$

$$(hg^{-1})^{-1}d(hg^{-1}) = (Ad g)(h^{-1}dh - g^{-1}dg) \quad (5.35)$$

$$(g^{-1})^{-1}d(g^{-1}) = -(Ad g)(g^{-1}dg) \quad (5.36)$$

$$(hgh^{-1})^{-1}d(hgh^{-1}) = (Ad h) \left((Ad g^{-1} - 1)(h^{-1}dh) + g^{-1}dg \right). \quad (5.37)$$

All of these follow from straightforward computations. We will derive only the last one.

$$\begin{aligned} (hgh^{-1})^{-1}d(hgh^{-1}) &= hg^{-1}h^{-1} \left((dh)gh^{-1} + h(dg)h^{-1} - hg(h^{-1}dh)h^{-1} \right) \\ &= h \left(g^{-1}(h^{-1}dh)g + g^{-1}dg - h^{-1}dh \right) h^{-1} \\ &= (Ad h) \left((Ad g^{-1} - 1)(h^{-1}dh) + g^{-1}dg \right). \end{aligned}$$

This proves (5.37).

For the derivation of (5.26) to (5.29) one need only note that $Ad g$ preserves all L^p norms. We see then that (5.26) follows from the identity (5.34)

together with the inequalities $\|gh - I_V\|_2 = \|(g - I_V)h + h - I_V\|_2 \leq \|g - I_V\|_2 + \|h - I_V\|_2$. (5.27) follows from (5.36) along with $\|g^{-1} - I_V\|_2 = \|I_V - g\|_2$. (5.29) follows from (5.37). For the proof of (5.28) we can compute that

$$\begin{aligned}\rho_p(gk, hk) &= \|(Ad k^{-1})(g^{-1}dg - h^{-1}dh)\|_p + \|gk - hk\|_2 \\ &= \|g^{-1}dg - h^{-1}dh\|_p + \|g - h\|_2 = \rho_p(g, h).\end{aligned}$$

The derivation of the simple properties (5.26) to (5.29) relied on the fact that multiplication by $Ad g$ preserves L^p norms. By contrast, multiplication by $Ad g$ does not preserve the H_a norms. The inequalities for ρ_a will depend for their proofs on the multiplier bounds of Proposition 5.9.

Apply (5.13) with $b = a$ and the appropriate choice of u to the identities (5.34), (5.36), (5.37) to find

$$\rho_a(hg, e) = \|(hg)^{-1}d(hg)\|_{H_a} + \|hg - I_V\|_2 \quad (5.38)$$

$$\begin{aligned}&\leq \|g^{-1}dg\|_{H_a} + (1 + c_1\|g^{-1}dg\|_3)\|h^{-1}dh\|_{H_a} + \|g - I_V\|_2 + \|h - I_V\|_2 \\ \rho_a(g^{-1}, e) &= \|(g^{-1})^{-1}d(g^{-1})\|_{H_a} + \|g^{-1} - I_V\|_2\end{aligned} \quad (5.39)$$

$$\begin{aligned}&\leq (1 + c_1\|g^{-1}dg\|_3)\|g^{-1}dg\|_{H_a} + \|g - I_V\|_2 \\ \rho_a(gk, hk) &= \|(Ad k^{-1})(g^{-1}dg - h^{-1}dh)\|_{H_a} + \|gk - hk\|_2 \\ &\leq (1 + c_1\|k^{-1}dk\|_3)\|g^{-1}dg - h^{-1}dh\|_{H_a} + \|g - h\|_2\end{aligned} \quad (5.40)$$

$$\begin{aligned}\rho_a(hgh^{-1}, e) &= \|(hgh^{-1})^{-1}d(hgh^{-1})\|_{H_a} + \|hgh^{-1} - I_V\|_2 \\ &\leq \left(1 + c_1\|h^{-1}dh\|_3\right)\|(Ad g^{-1} - 1)(h^{-1}dh) + g^{-1}dg\|_{H_a} + \|g - I_V\|_2 \\ &\leq \left(1 + c_1\|h^{-1}dh\|_3\right)\left(\|(Ad g^{-1} - 1)(h^{-1}dh)\|_{H_a} + \|g^{-1}dg\|_{H_a}\right) + \|g - I_V\|_2.\end{aligned} \quad (5.41)$$

Each of these inequalities holds for all $a \in [0, 1]$. However we now wish to estimate several of the L^3 norms that occur in these inequalities by an H_a norm. By Sobolev we have $\|u\|_3 \leq \kappa_3\|u\|_{H_{1/2}} \leq \kappa_3\|u\|_{H_a}$ if $a \geq 1/2$. Thus we may dominate the factors $(1 + c_1\|g^{-1}dg\|_3)$ by $(1 + c_2\|g^{-1}dg\|_{H_a}) \leq 1 + c_2\rho_a(g, e)$ in (5.38) and (5.39) and dominate the factor $1 + c_1\|k^{-1}dk\|_3$ by $1 + c_2\rho_a(k, e)$ in (5.40). This completes the proof of (5.30) through (5.32). The inequality (5.33) follows from (5.41) if one takes into account that $\rho_a(g, e) = \|g^{-1}dg\|_{H_a} + \|g - I_V\|_2$. ■

Proof of Lemma 5.12. $\mathcal{G}_{1,p}$ is closed under multiplication and inversion by (5.26) and (5.27). The identity (5.28) shows that ρ_p is a right invariant

metric. The right invariance of ρ_p ensures that right multiplication is continuous in this metric. We will use (5.29) later to show that left multiplication is also continuous.

\mathcal{G}_{1+a} is closed under pointwise multiplication and inversion by (5.30) and (5.31). Although ρ_a is not right invariant, the topology induced by the metric ρ_a is invariant under right multiplication, as follows immediately from (5.32). Hence right multiplication is continuous in \mathcal{G}_{1+a} . ■

To show that $\mathcal{G}_{1,p}$ and \mathcal{G}_{1+a} are topological groups it still needs to be shown that left multiplication and inversion are continuous. These will be proven in the next sections.

5.4 $\mathcal{G}_{1,p}$ is a topological group

Theorem 5.14 *For $2 \leq p < \infty$ multiplication and inversion are continuous in $\mathcal{G}_{1,p}$. $\mathcal{G}_{1,p}$ is a topological group. For any index $q \in [2, \infty)$ the map*

$$\mathcal{G}_{1,p} \ni g \mapsto (\text{Ad } g : L^q \rightarrow L^q) \quad (5.42)$$

is a strongly continuous representation of $\mathcal{G}_{1,p}$ into isometries of $L^q(M; \Lambda^1 \otimes \mathfrak{k})$. Moreover if $p > 3$ then the representation is norm continuous.

The proof depends on the following lemma.

Lemma 5.15 *(Strong and norm continuity on L^q .) Let $2 \leq q < \infty$.*

a) Let $u \in L^q(M; \Lambda^1 \otimes \mathfrak{k})$ and let $\epsilon > 0$. Then there exists $\delta > 0$, depending on ϵ and u , such that, for any function $g : M \rightarrow K$, one has

$$\|(\text{Ad } g)u - u\|_q < \epsilon \quad \text{whenever} \quad \|g - I_V\|_2 < \delta. \quad (5.43)$$

b) Let $p > 3$. Given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|\text{Ad } g - 1\|_{L^q \rightarrow L^q} < \epsilon \quad \text{whenever} \quad \|g^{-1}dg\|_p + \|g - I_V\|_2 < \delta. \quad (5.44)$$

Proof. If, for some $x \in M$, $|u(x)|_{\Lambda^1 \otimes \mathfrak{k}} \leq \lambda$ then $|(\text{Ad } g(x) - I_V)v(x)|_{\Lambda^1 \otimes \mathfrak{k}} \leq 2\lambda|g(x) - I_V|_{\text{End } V}$. Here, $|\cdot|_{\Lambda^1 \otimes \mathfrak{k}}$ refers to the Euclidean norm on $\Lambda^1 \otimes \mathfrak{k}$. Thus

$$\begin{aligned} \|(\text{Ad } g - 1)u\|_q &\leq \|(\text{Ad } g - 1)\chi_{|u|>\lambda}u\|_q + 2\lambda\|g - I_V\|_q \\ &\leq 2\|\chi_{|u|>\lambda}u\|_q + 2\lambda \cdot 2^{1-(2/q)}\|g - I_V\|_2^{2/q}. \end{aligned}$$

Hence, given $u \in L^q$ and $\epsilon > 0$, choose λ so large that the first term is at most $\epsilon/2$ and then choose $\delta > 0$ so small that the second term is also less than $\epsilon/2$ when $\|g - I_V\|_2 < \delta$. This proves (5.43). Notice that the restraint on δ depends on λ , which depends on u and not just on $\|u\|_q$.

For the proof of (5.44) assume that $p > 3$ and observe first that $\|d(g - I_V)\|_p = \|dg\|_p = \|g^{-1}dg\|_p$. Since $(1/p) - (1/3) < 0$, Sobolev's inequality shows that $\|g^{-1}dg\|_p + \|g - I_V\|_2$ controls $\|g - I_V\|_\infty$ and therefore also $\|Ad g - 1\|_\infty$, which is the norm of $Ad g - 1$ as an operator on $L^q(M; \Lambda^1 \otimes \mathfrak{k})$.

(Notice that adding $\|g^{-1}dg\|_3$ to the norm in (5.43) will not help to dominate $\|g - I_V\|_\infty$ because $(1/3) - (1/3) = 0$.) ■

Proof of Theorem 5.14. Since the metric ρ_p is right invariant, right translation is a homeomorphism of $\mathcal{G}_{1,p}$, and therefore a neighborhood of a point g_0 can be represented in the form $U(g_0) = \{\alpha g_0 : \rho_p(\alpha, e) < \delta\}$. If also $h_0 \in \mathcal{G}_{1,p}$ and we take $V = \{\beta h_0 : \rho_p(\beta, e) < \delta\}$ as a neighborhood of h_0 then a product of points in these neighborhoods may be written $gh = (\alpha g_0)(\beta h_0) = \gamma g_0 h_0$ where $\gamma = \alpha(g_0 \beta g_0^{-1})$. By (5.29) we have

$$\rho_p(g_0 \beta g_0^{-1}, e) \leq \rho_p(\beta, e) + \|(Ad \beta - 1)g_0^{-1}dg_0\|_p. \quad (5.45)$$

By Lemma 5.15, the entire right hand side of (5.45) can be made small by choosing δ small. Thus, in view of (5.26), given $\epsilon > 0$ there exists $\delta > 0$ such that $\rho_p(\gamma, e) < \epsilon$. Multiplication is therefore jointly continuous. In particular left translations are homeomorphisms of $\mathcal{G}_{1,p}$. Since a right translate of a basic neighborhood N of the identity by an element $g \in \mathcal{G}_{1,p}$ is carried by inversion into a left translate by g^{-1} of the inverse N^{-1} , which is itself open by (5.27), it follows that inversion is continuous. Thus multiplication and inversion are continuous and so $\mathcal{G}_{1,p}$ is a topological group.

For the proof of strong continuity of the representation $\mathcal{G}_{1,p} \ni g \mapsto Ad g$ on $L^q(M; \Lambda^1 \otimes \mathfrak{k})$ it suffices to show that for fixed $u \in L^q(M; \Lambda^1 \otimes \mathfrak{k})$ the map $\mathcal{G}_{1,p} \ni g \mapsto (Ad g)u$ is continuous into L^q at $g = I_V$. But $\|g - I_V\|_2 \leq \rho_p(g, I_V)$ by the definition (5.9). The strong continuity now follows from Part a) of Lemma 5.15. If $p > 3$ then Part b) of Lemma 5.15 shows that the map $g \mapsto Ad g$ is actually norm continuous on L^q . ■

5.5 \mathcal{G}_{1+a} is a topological group if $a \geq 1/2$

Theorem 5.16 (\mathcal{G}_{1+a} is a topological group) *If $1/2 \leq a \leq 1$ then multiplication and inversion are continuous in \mathcal{G}_{1+a} . In particular \mathcal{G}_{1+a} is a topological*

group.

The critical case $a = 1/2$ will be the most delicate case in this theorem. The proof depends on the following strong continuity lemma.

Lemma 5.17 (*Strong continuity of $\mathcal{G}_{1,3}$ on H_b .*) For $0 \leq b \leq 1$ the map

$$\mathcal{G}_{1,p} \ni g \mapsto (Ad\,g : H_b \rightarrow H_b) \quad (5.46)$$

is a strongly continuous representation of $\mathcal{G}_{1,p}$ into bounded operators on H_b if $p = 3$. If $p > 3$ and M has finite volume then the representation is norm continuous.

Proof. We already know from (5.13) that $Ad\,g$ is bounded on H_b . For the proof of strong continuity suppose first that $p = 3$. Let $u \in H_b$ and $\epsilon > 0$ be given. We need to show that there exists $\delta > 0$, depending on ϵ and u such that

$$\|(Ad\,g - 1)u\|_{H_b} < \epsilon \quad \text{whenever} \quad \|g^{-1}dg\|_3 + \|g - I_{\mathcal{V}}\|_2 < \delta. \quad (5.47)$$

Choose $\delta_1 \in (0, 3/2)$ and let $p_1 = 3/\delta_1$. Pick $\lambda < \infty$ such that $\|\chi_{[\lambda,\infty)}(D)u\|_{H_b} < \epsilon/6$. Let $v = \chi_{[0,\lambda)}(D)u$ and $w = \chi_{[\lambda,\infty)}(D)u$. Then $u = v + w$ is an orthogonal decomposition of u in H_b . Moreover $\|w\|_{H_b} < \epsilon/6$ and $\|v\|_{H_{b+\delta_1}} \leq \lambda^{\delta_1} \|v\|_{H_b} \leq \lambda^{\delta_1} \|u\|_{H_b}$ by the spectral theorem. In view of (5.15) and (5.14), we have

$$\begin{aligned} \|(Ad\,g - 1)u\|_{H_b} &\leq \|(Ad\,g - 1)v\|_{H_b} + \|(Ad\,g - 1)w\|_{H_b} \\ &\leq \left(\kappa_{\delta_1} \|Ad\,g - 1\|_{p_1} + c_1 \|g^{-1}dg\|_3 \right) \|v\|_{H_{b+\delta_1}} \\ &\quad + \left(\|Ad\,g - 1\|_{\infty} + c_1 \|g^{-1}dg\|_3 \right) \|w\|_{H_b} \\ &\leq \left(\kappa_{\delta_1} \|Ad\,g - 1\|_{p_1} + c_1 \|g^{-1}dg\|_3 \right) \lambda^{\delta_1} \|v\|_{H_b} + \left(2 + c_1 \|g^{-1}dg\|_3 \right) \epsilon/6. \end{aligned}$$

Since $|g(x) - I_{\mathcal{V}}|_{op} \leq 2$ we have the pointwise operator bound $|g(x) - I_{\mathcal{V}}|_{op}^{p_1} \leq 2^{p_1-2} |g(x) - I_{\mathcal{V}}|_{op}^2$ for $2 \leq p_1 < \infty$. Therefore $\|g - I_{\mathcal{V}}\|_{p_1} \leq 2^{1-(2/p_1)} \|g - I_{\mathcal{V}}\|_2^{2/p_1}$. Further, since g is unitary (or orthogonal), $\|Ad\,g - I_{End\,\mathcal{V}}\|_{p_1} \leq 2\|g - I_{\mathcal{V}}\|_{p_1}$. Hence

$$\begin{aligned} \|(Ad\,g - 1)u\|_{H_b} &\leq \left(2^{2-(2/p_1)} \kappa_{\delta_1} \|g - I_{\mathcal{V}}\|_2^{2/p_1} + c_1 \|g^{-1}dg\|_3 \right) \lambda^{\delta_1} \|u\|_{H_b} \\ &\quad + \left(2 + c_1 \|g^{-1}dg\|_3 \right) \epsilon/6. \end{aligned} \quad (5.48)$$

Thus if δ is chosen small enough in (5.47), and in particular $c_1\delta < 1$, then the last term in (5.48) will be at most $\epsilon/2$ while the first term on the right side can also be made less than $\epsilon/2$. This proves strong continuity at the identity element of $\mathcal{G}_{1,3}$.

In case $p > 3$ we can simply use the crude estimate (5.14) in place of the deeper estimate (5.15) because $\|g - I_V\|_2$, together with $\|dg\|_p$, which equals $\|g^{-1}dg\|_p$, control $\|g - I_V\|_\infty$ and therefore also $\|Ad g - 1\|_\infty$. Thus, given $\epsilon > 0$, the entire coefficient of $\|u\|_{H_b}$ on the right side of (5.14) will be less than ϵ if $\rho_p(g, e) < \delta$ for small enough δ . Here we are using the finite volume of M only to dominate the L^3 norm of $g^{-1}dg$ by the L^p norm. This proves norm continuity of the representation $g \mapsto Ad g$ on H_b . ■

Proof of Corollary 5.5. If $1/p = 1/2 - a/3$ then \mathcal{G}_{1+a} embeds continuously and homomorphically into $\mathcal{G}_{1,p}$ for $a \in [1/2, 1]$. By Lemma 5.17 the representation $\mathcal{G}_{1+a} \ni g \mapsto Ad g$ on H_b is therefore strongly continuous for $a = 1/2$ and norm continuous for $a \in (1/2, 1]$. It is not necessary to specify that M have finite volume to prove the norm continuity because $H_a \subset H_{1/2}$ if $a > 1/2$ and therefore $\mathcal{G}_{1+a} \subset \mathcal{G}_{3/2}$, while $\mathcal{G}_{3/2}$ controls $\|g^{-1}dg\|_3$, whose control was the only reason for requiring M to have finite volume in the last two lines of the previous proof. ■

Proof of Theorem 5.16. We need to prove that multiplication on the left is continuous. Surprisingly, this soft sounding assertion seems to require use of the complex interpolation methods that underlie the multiplier bounds of Section 5.2, at least in the critical case $a = 1/2$. Otherwise the proof is similar to that for $\mathcal{G}_{1,p}$.

Since right translation is a homeomorphism of \mathcal{G}_{1+a} , a neighborhood of a point g_0 can be represented in the form $U(g_0) = \{\alpha g_0 : \rho_a(\alpha, e) < \delta\}$. If also $h_0 \in \mathcal{G}_{1+a}$ and $V = \{\beta h_0 : \rho_a(\beta, e) < \delta\}$ is a neighborhood of h_0 then a product of points in these neighborhoods may be written $gh = (\alpha g_0)(\beta h_0) = \gamma g_0 h_0$ where $\gamma = \alpha(g_0 \beta g_0^{-1})$. We need to show that γ is close to the identity if α and β are. Recall that g_0 and h_0 are fixed. By (5.33) we have

$$\rho_a(g_0 \beta g_0^{-1}, e) \leq (1 + c_1 \|g_0^{-1} d g_0\|_3) \left(\rho_a(\beta, e) + \|(Ad \beta^{-1} - 1) g_0^{-1} d g_0\|_{H_a} \right). \quad (5.49)$$

When $\rho_a(\beta, e)$ is small, so is $\rho_a(\beta^{-1}, e)$, by (5.31) and therefore, by Lemma 5.17, the entire right hand side of (5.49) can be made small by choosing δ

small. Thus, in view of (5.30), given $\epsilon > 0$ there exists $\delta > 0$ such that $\rho_a(\gamma, e) < \epsilon$. Multiplication is therefore jointly continuous. In particular left translations are homeomorphisms of \mathcal{G}_{1+a} . Since a right translate of a neighborhood N of the identity by an element $g \in \mathcal{G}_{1+a}$ is carried by inversion into a left translate by g^{-1} of the inverse N^{-1} , which is itself open by (5.31), it follows that inversion is continuous. ■

5.6 Completeness

Lemma 5.18 (*Completeness*) \mathcal{G}_{1+a} is complete for $1/2 \leq a \leq 1$.

Proof. Suppose that $\{g_n\}_{n=0}^\infty$ is a Cauchy sequence in \mathcal{G}_{1+a} with $1/2 \leq a \leq 1$. Choose a subsequence, denoted again the same way, such that

$$\rho_a(g_n, g_{n-1}) \leq 1/2^n, \quad n = 1, 2, \dots \quad (5.50)$$

Let

$$u_n = g_n^{-1}dg_n - g_{n-1}^{-1}dg_{n-1}, \quad n = 1, 2, \dots \quad (5.51)$$

Then

$$\|u_n\|_{H_a} \leq 1/2^n. \quad (5.52)$$

From (5.51) we see that

$$g_n^{-1}dg_n = \sum_{k=1}^n u_k + g_0^{-1}dg_0. \quad (5.53)$$

So the sequence in (5.53) is convergent in H_a . There exists a unique element $v \in H_a$ such that

$$v = \lim_{n \rightarrow \infty} \sum_{k=1}^n u_k + g_0^{-1}dg_0 \quad (\text{convergence in } H_a \text{ norm}). \quad (5.54)$$

To construct the desired limit $g \in \mathcal{G}_{1+a}$ observe first that

$$\begin{aligned} \|dg_n - dg_k\|_2 &= \|g_n^{-1}(dg_n - dg_k)\|_2 \\ &= \|g_n^{-1}dg_n - g_k^{-1}dg_k + (g_k^{-1} - g_n^{-1})dg_k\|_2 \\ &\leq \|g_n^{-1}dg_n - g_k^{-1}dg_k\|_2 + \|(I_V - g_n^{-1}g_k)(g_k^{-1}dg_k)\|_2. \end{aligned} \quad (5.55)$$

The first term on the right side of (5.55) goes to zero as $n, k \rightarrow \infty$ because $\sum_1^\infty \|u_k\|_{H_a} < \infty$. Since $g_k^{-1}dg_k$ converges in H_a it also converges in $L^2(M)$. And, since $\|I_{\mathcal{V}} - g_n^{-1}g_k\|_2 = \|g_n - g_k\|_2 \rightarrow 0$, the factor $I_{\mathcal{V}} - g_n^{-1}g_k$ converges to zero in measure and boundedly. Therefore the second term in line (5.55) also goes to zero. Thus $\|dg_n - dg_k\|_2 \rightarrow 0$ and $\|g_n - g_k\|_2 \rightarrow 0$. Hence there exists a function $g \in H_1(M; \text{End } \mathcal{V})$ such that $\|g_n - g\|_{H_1} \rightarrow 0$. Partly reversing the argument in (5.55) we find

$$\begin{aligned} \|g_n^{-1}dg_n - g^{-1}dg\|_2 &= \|dg_n - g_n g^{-1}dg\|_2 \\ &\leq \|dg_n - dg\|_2 + \|(I_{\mathcal{V}} - g_n g^{-1})dg\|_2. \end{aligned}$$

The first term on the right goes to zero, as we have just seen. Now $\|dg\|_2 < \infty$ while $(I_{\mathcal{V}} - g_n g^{-1})$ converges to zero in L^2 and therefore in measure. Since this factor is bounded by 2 we can now apply the dominated convergence theorem to conclude that the second term goes to zero also.

We already know that $g_n^{-1}dg_n$ converges to v in the H_a sense. Since it also converges to $g^{-1}dg$ in L^2 it follows that $v = g^{-1}dg$. In particular $g \in \mathcal{G}_{1+a}$ and $\rho_a(g_n, g) \rightarrow 0$. This completes the proof of Lemma 5.18 and Theorem 5.3. ■

Lemma 5.19 *For $2 \leq p < \infty$ the groups $\mathcal{G}_{1,p}$ are complete.*

Proof. The proof is similar to the proof for the groups \mathcal{G}_{1+a} : Dropping to a subsequence such that $\rho_p(g_n, g_{n-1}) \leq 2^{-n}$ we see that $\{g_n^{-1}dg_n\}$ is a Cauchy sequence in L^p . The inequality $\|dg_n - dg_k\|_p \leq \|g_n^{-1}dg_n - g_k^{-1}dg_k\|_p + \|(I_{\mathcal{V}} - g_n^{-1}g_k)(g_k^{-1}dg_k)\|_p$ now shows that $\|dg_n - dg_k\|_p \rightarrow 0$ by the same argument used after (5.55), which uses our hypothesis in this lemma that $\|g_n - g_k\|_2 \rightarrow 0$. Therefore the sequence $\{g_n\}$ is itself a Cauchy sequence in the Sobolev space

$$\left\{ f : M \rightarrow \text{End } \mathcal{V} \mid \int_M \sum_{j=1}^3 |\partial_j f(x)|^p dx + \int_M |f(x)|^2 dx < \infty \right\} \quad (5.56)$$

and so converges to some function g which takes its values in K (because there is a subsequence which converges a.e.) and such that $dg \in L^p(M; \Lambda^1 \otimes \text{End } \mathcal{V})$. Now the inequality $\|g_n^{-1}dg_n - g^{-1}dg\|_p \leq \|dg_n - dg\|_p + \|(I_{\mathcal{V}} - g_n g^{-1})dg\|_p$ shows that $g_n^{-1}dg_n$ converges in L^p to $g^{-1}dg$. Thus $g \in \mathcal{G}_{1,p}$ and $\rho_p(g_n, g) \rightarrow 0$. This completes the proof of Lemma 5.19 and of Theorem 5.2. ■

Remark 5.20 We are only interested in the groups $\mathcal{G}_{1,p}$ for $p \geq 2$. But the for $1 < p < 2$ Theorem 5.2 also holds if one uses the right invariant metric defined by $\rho_p(g, e) = \|g^{-1}dg\|_p + \|g - I_V\|_p$. Proofs are similar.

Remark 5.21 (Differentiable structures, Hilbert and Banach Lie groups) \mathcal{G}_{1+a} is a Hilbert Lie group if $a > 1/2$ and $\mathcal{G}_{1,p}$ is a Banach Lie group if $p > 3$. Both assertions follow from the fact that the metric on the gauge group controls $\|g - I_V\|_\infty$: Let

$$\begin{aligned}\mathcal{L}_a &\equiv \{\alpha : M \rightarrow \mathfrak{k} \mid \|\alpha\|_{H_{1+a}} < \infty\} \text{ and} \\ \mathcal{L}_p &\equiv \{\alpha : M \rightarrow \mathfrak{k} \mid \|\alpha\|_{H_{1,p}} < \infty\},\end{aligned}$$

where we have written $\|\alpha\|_{H_{1,p}} = \|D\alpha\|_p$. If $a > 1/2$ then $\|\alpha\|_\infty \leq \text{const.} \|\alpha\|_{H_{1+a}}$. Consequently \mathcal{L}_a is closed under the pointwise Lie bracket operation and is a Hilbert Lie algebra. Similarly if $p > 3$ then $\|\alpha\|_\infty \leq \text{const.} \|\alpha\|_{H_{1,p}}$ and so \mathcal{L}_p is a Banach Lie algebra. The exponential map $\alpha \mapsto (g : x \mapsto \exp \alpha(x))$ maps a neighborhood of zero in the Lie algebra \mathcal{L}_a , resp. \mathcal{L}_p , onto a neighborhood of I_V in \mathcal{G}_{1+a} , respectively $\mathcal{G}_{1,p}$, which follows from the fact that in the metric on the gauge group there is a neighborhood of I_V contained in $\{g : \|g - I_V\|_\infty < \epsilon\}$, as may be seen from the proof of Part b) of Lemma 5.15 and the continuity of the injection $\mathcal{G}_{1+a} \rightarrow \mathcal{G}_{1,p}$ for $p^{-1} = 2^{-1} - (a/3)$. Thus for small ϵ one can use the known surjectivity of the exponential map in K to prove the existence of a function α such that $g(x) = \exp \alpha(x)$ for all $x \in M$. If ϕ is the inverse of the exponential map on a small neighborhood of the identity in K then the formula $\alpha(x) = \phi(g(x))$ transfers regularity of g to the same Sobolev regularity of α . Thus the tangent space at the identity of \mathcal{G}_{1+a} , resp. $\mathcal{G}_{1,p}$ can be identified with \mathcal{L}_a , resp. \mathcal{L}_p . We leave to the reader to verify that the topologies on these two classes of groups, given respectively by the metrics (5.8) and (5.9), agree with those induced by the norms on the Lie algebras.

This construction of a differentiable structure breaks down in case $a = 1/2$ or $p = 3$. It seems highly unlikely that in these critical cases there is a useful differentiable structure on $\mathcal{G}_{3/2}$ or on $\mathcal{G}_{1,3}$. The fact that $\mathcal{G}_{3/2}$ and $\mathcal{G}_{1,3}$ are actually topological groups (i.e. products and inversion are continuous) is thanks to our avoidance of the exponential map in Definitions (5.2) - (5.9). The Hilbert space $\mathcal{L}_{3/2}$ and Banach space $\mathcal{L}_{1,3}$ are not closed under Lie bracket. Nevertheless $\exp \mathcal{L}_{3/2} \subset \mathcal{G}_{3/2}$ and $\exp \mathcal{L}_{1,3} \subset \mathcal{G}_{1,3}$. It will be shown elsewhere that $\exp \mathcal{L}_{1,3}$ does not cover any neighborhood of the identity in

$\mathcal{G}_{1,3}$ if $K = SU(2)$. This strongly suggests that $\exp \mathcal{L}_{3/2}$ also does not cover any neighborhood of the identity in $\mathcal{G}_{3/2}$.

The group that we have denoted by $\mathcal{G}_{1,2}$ has been used by G. Dell'Antonio and D. Zwanziger, [6], to give a very pretty proof that every gauge orbit intersects $\{A : \int_M |A(x)|^2 dx < \infty\}$ at a point which minimizes this L^2 norm. M can be a d dimensional manifold. Their result illuminates the Gribov ambiguity.

6 The conversion group

In this section we will take M to be either all of \mathbb{R}^3 or the closure of a bounded, convex, open subset of \mathbb{R}^3 with smooth boundary.

6.1 The ZDS procedure

Definition 6.1 (Definition of g_ϵ .) Suppose that $C(\cdot)$ is a smooth solution to the augmented Yang-Mills heat equation (2.22) over $(0, T)$. Let $\epsilon \in (0, T)$ and define $g_\epsilon(t)$ to be the solution to the ODE, for each (suppressed) $x \in M$,

$$\frac{dg_\epsilon(t)}{dt} g_\epsilon(t)^{-1} = d^*C(t), \quad t \in (0, T), \quad g_\epsilon(\epsilon) = I_V. \quad (6.1)$$

Then $g_\epsilon \in C^\infty((0, T] \times M; K)$ because $d^*C(t, x)$ is smooth on $(0, T] \times M$.

The ZDS procedure for recovering a solution to the Yang-Mills heat equation (2.5) from a solution to the augmented equation (2.22) is outlined in the Introduction. Informally, the function $g(t)$ defined in (1.8) is the function $g_\epsilon(t)$ for $\epsilon = 0$. As $\epsilon \downarrow 0$, however, the functions $g_\epsilon(\cdot)$ lose smoothness in both space and time. This results from the strong singularity of $d^*C(t)$ at $t = 0$. In case $a = 1/2$ one has, typically, $d^*C(0) \in H_{-1/2}(M)$. In the next theorem we will show that as $\epsilon \downarrow 0$ the functions $g_\epsilon(\cdot)$ converge uniformly over $(0, T]$ as functions into the gauge group \mathcal{G}_{1+a} . Typically, a gauge function in \mathcal{G}_{1+a} is continuous on M if $1/2 < a < 1$ but not smooth. If $a = 1/2$ it need not even be continuous.

Theorem 6.2 (*The conversion group*) Let $1/2 \leq a < 1$ and $0 < T < \infty$. Assume that either $M = \mathbb{R}^3$ or is the closure of a bounded, convex, open set in \mathbb{R}^3 with smooth boundary. Let $A_0 \in H_a$. Suppose that $C(\cdot)$ is the strong

solution of the augmented equation (2.22) over $[0, T]$ constructed in Theorem 2.20 with initial value $C_0 = A_0$. Assume that $C(\cdot)$ has finite strong a -action. (This is automatic for $a > 1/2$.) Define g_ϵ as in (6.1). Then $g_\epsilon(t) \in \mathcal{G}_{1+a}$ for each $t \in (0, T]$. Further,

- a) $g_\epsilon : (0, T] \rightarrow \mathcal{G}_{1+a}$ is continuous.
- b) There is a unique continuous function

$$g : [0, T] \rightarrow \mathcal{G}_{1+a} \quad (6.2)$$

such that

$$\lim_{\epsilon \downarrow 0} \sup_{0 < t \leq T} \rho_a(g_\epsilon(t), g(t)) = 0. \quad (6.3)$$

c)

$$g(0, x) = I_{\mathcal{V}} \quad \forall x \in M. \quad (6.4)$$

d) The function $h(t) \equiv g(t)^{-1}dg(t)$ is continuous on $[0, T]$ into H_a and

$$h(0) = 0. \quad (6.5)$$

e) For any time $\tau \in (0, T)$, the function

$$k(t) \equiv g(t)g(\tau)^{-1} \quad (6.6)$$

is in $C^\infty((0, T] \times M; K)$. Moreover $\lim_{t \downarrow 0} k(t) = g(\tau)^{-1}$, with convergence in the sense of the metric group \mathcal{G}_{1+a} .

Remark 6.3 (Strategy) The proof of the theorem will proceed in three steps. It will first be proven that the functions g_ϵ converge in a relatively weak sense, namely as functions into $L^p(M; \text{End } \mathcal{V})$ for all $p < \infty$. This will then be used to show that they are bounded as functions into the metric group $\mathcal{G}_{1,q}$ for q appropriately related to a . This in turn will then be used to prove the strong sense of convergence asserted in Theorem 6.2, namely as functions into the metric group \mathcal{G}_{1+a} .

Remark 6.4 (Smoothness) The singularity in $d^*C(t)$ as $t \downarrow 0$ reflects itself in a lack of smoothness of $g(t, \cdot)$ for each $t > 0$, not just “near” $t = 0$. We will see in Section 7 how this then reflects itself in a lack of smoothness of $A(t, \cdot)$ for each $t > 0$. The function $A(t, \cdot)$ need not even be in $H_1(M)$ for each $t > 0$. On the other hand Part e) of the theorem shows that the singularity disappears from ratios. This lies behind the assertion in Theorems 2.10 and 2.11 that the solution is gauge equivalent to a strong solution, which is in fact C^∞ for some time. See Theorem 7.1 for a precise statement.

6.2 g estimates

We will prove in this section that the functions g_ϵ converge as $\epsilon \downarrow 0$, but in a much weaker sense than that asserted in Theorem 6.2.

Lemma 6.5 *Let $2 \leq p < \infty$. Under the hypotheses of Theorem 6.2 the functions $(0, T] \ni t \mapsto g_\epsilon(t)$ are continuous functions into $L^p(M; \text{End } \mathcal{V})$. There is a continuous function $g : [0, T] \rightarrow L^p(M; \text{End } \mathcal{V})$ to which the functions g_ϵ converge, uniformly over $(0, T]$. That is,*

$$\sup_{0 < t \leq T} \|g_\epsilon(t) - g(t)\|_p \rightarrow 0 \quad \text{as } \epsilon \downarrow 0. \quad (6.7)$$

Moreover $g(0, x) = I_{\mathcal{V}}$ for all $x \in M$. For each $t \in [0, T]$, $g(t, x)$ lies in K for almost all $x \in M$. If $a > 1/2$ then (6.7) holds also for $p = \infty$. In this case $g(\cdot, \cdot)$ is continuous on $[0, T] \times M$ into K .

Remark 6.6 In the critical case $a = 1/2$ it seems doubtful that the function $x \mapsto g(t, x)$ need be continuous on M for any fixed $t > 0$. The strong sense of convergence asserted in Theorem 6.2, Part b) does not assure that $g(t, \cdot)$ is continuous for fixed $t > 0$ because the metric on $\mathcal{G}_{3/2}$ does not control the supremum norm on differences $g_\epsilon(t, \cdot) - g_\delta(t, \cdot)$ in $\text{End } \mathcal{V}$. Thus in the critical case there may be a bundle change for each $t > 0$.

Proof of Lemma 6.5. Let $\phi(t) = d^*C(t)$ as in (4.4). The differential equation (6.1) implies, for $0 < \delta \leq \epsilon$ and for each point $x \in M$, that $g_\delta(t) = g_\epsilon(t)g_\delta(\epsilon)$ for all $t \in (0, T]$. Hence $g_\delta(t) - g_\epsilon(t) = g_\epsilon(t)(g_\delta(\epsilon) - I_{\mathcal{V}})$. Moreover, by (6.1) one has $g'_\delta(t, x) = \phi(t, x)g_\delta(t, x)$ and therefore $\|g_\delta(\epsilon, x) - I_{\mathcal{V}}\|_{op} = \|\int_\delta^\epsilon \phi(t, x)g_\delta(t, x)dt\|_{op} \leq \int_\delta^\epsilon \|\phi(t, x)\|_{op}dt$. Hence, for each point $x \in M$ we have

$$\|g_\delta(t, x) - g_\epsilon(t, x)\|_{op} = \|g_\delta(\epsilon, x) - I_{\mathcal{V}}\|_{op} \leq \int_\delta^\epsilon \|\phi(s, x)\|_{op}ds, \quad 0 < t \leq T.$$

Therefore, for $2 \leq p < \infty$, one has

$$\|g_\delta(t) - g_\epsilon(t)\|_p = \|g_\delta(\epsilon) - I_{\mathcal{V}}\|_p \leq \int_\delta^\epsilon \|\phi(s)\|_p ds \quad \text{for } 0 < t \leq T. \quad (6.8)$$

It follows from the integrability $\int_0^T \|\phi(s)\|_p ds < \infty$, proven in (4.98) (for $6 \leq p < \infty$), and implied by (4.36) (for $p = 2$), and which holds therefore for all $p \in [2, \infty)$ by interpolation, that

$$\sup_{0 < \delta \leq \epsilon} \sup_{0 < t < T} \|g_\delta(t) - g_\epsilon(t)\|_p \rightarrow 0, \quad \text{as } \epsilon \downarrow 0. \quad (6.9)$$

Hence the functions g_ϵ converge uniformly on $(0, T]$ to a continuous function $g : (0, T] \rightarrow L^p$. Fix ϵ in (6.8) and let $\delta \downarrow 0$ to find

$$\|g(\epsilon) - I_{\mathcal{V}}\|_p \leq \int_0^\epsilon \|\phi(s)\|_p ds, \quad (6.10)$$

which goes to zero as $\epsilon \downarrow 0$. Therefore, by defining $g(0, x) = I_{\mathcal{V}}$ for all $x \in M$, one obtains a continuous function $g : [0, T) \rightarrow L^p(M; \text{End } \mathcal{V})$ for $2 \leq p < \infty$.

In case $a > 1/2$ the inequality (4.92) shows that we can simply replace the L^p norm in (6.8) - (6.10) by the L^∞ norm. In this case the convergence in (6.7) is uniform in both space and time. $g(\cdot, \cdot)$ is therefore continuous.

It will be useful to record here the observation that

$$\sup_{0 < \delta \leq t} \|g_\delta(t) - I_{\mathcal{V}}\|_p \rightarrow 0 \quad \text{as } t \downarrow 0, \quad 2 \leq p < \infty, \quad (6.11)$$

which follows from (6.8) with $\epsilon = t$, namely, $\|g_\delta(t) - I_{\mathcal{V}}\|_p \leq \int_0^t \|\phi(s)\|_p ds$. ■

6.3 The vertical projection

Notation 6.7 In the Hilbert space $L^2(M; \Lambda^1 \otimes \mathfrak{k})$ the subspace

$$\mathcal{H} \equiv \{\omega \in L^2(M; \Lambda^1 \otimes \mathfrak{k}) : d^*\omega = 0\} \quad (6.12)$$

is a closed subspace of $L^2(M; \Lambda^1 \otimes \mathfrak{k})$ because d^* is a closed operator. If $M \neq \mathbb{R}^3$ then d^* refers to the maximal operator in the case of Dirichlet boundary conditions and to the minimal operator in the case of Neumann boundary conditions. \mathcal{H} is the horizontal subspace for the Coulomb connection at the connection form zero. Denote by \mathcal{H}^\perp its orthogonal complement and by P^\perp the orthogonal projection onto \mathcal{H}^\perp .

The next lemma concerns the well known projection onto the exact 1-forms in the Hodge decomposition. We are going to carry out some of the details because of possible technical problems associated to Neumann boundary conditions.

Lemma 6.8 *The restriction of P^\perp to $H_1(M; \Lambda^1 \otimes \mathfrak{k})$ is a bounded operator from H_1 into H_1 . Moreover*

$$dP^\perp\omega = 0 \quad \forall \omega \in L^2(M; \Lambda^1 \otimes \mathfrak{k}). \quad (6.13)$$

$$d^*P^\perp\omega = d^*\omega \quad \forall \omega \in \mathcal{D}(d^*). \quad (6.14)$$

Proof. If $\omega \in L^2$ and $u \in \mathcal{D}(d^*)$ then $d^*u \in \mathcal{D}(d^*)$ by [2, Proposition 3.5] and $d^*d^*u = 0$. Therefore $d^*u \in \mathcal{H}$. Hence, for any 1-form $\omega \in L^2(M; \Lambda^1 \otimes \mathfrak{k})$ we have $(P^\perp\omega, d^*u) = 0$ for all $u \in \mathcal{D}(d^*)$. Since d and d^* are closed operators it follows that $P^\perp\omega \in \mathcal{D}(d)$ and $dP^\perp\omega = 0$ for all $\omega \in L^2$. This proves (6.13). Now $\omega - P^\perp\omega \in \mathcal{H} \subset \mathcal{D}(d^*)$. So if $\omega \in \mathcal{D}(d^*)$ then $P^\perp\omega \in \mathcal{D}(d^*)$ and $d^*\omega - d^*P^\perp\omega = 0$. This proves (6.14). From the Gaffney-Friedrichs inequality, [2, Equ. (2.22)], we then have, for $\omega \in H_1$,

$$\begin{aligned} \|P^\perp\omega\|_{H_1}^2 &\leq 2\left(\|d(P^\perp\omega)\|_2^2 + \|d^*(P^\perp\omega)\|_2^2 + \|P^\perp\omega\|_2^2\right) \\ &= 2\left(\|d^*\omega\|_2^2 + \|P^\perp\omega\|_2^2\right) \leq 2\left(\|d^*\omega\|_2^2 + \|\omega\|_2^2\right) \\ &\leq 2\|\omega\|_{H_1}^2. \end{aligned}$$

Thus $P^\perp : H_1 \rightarrow H_1$ is bounded and (6.13) and (6.14) hold. ■

Remark 6.9 It is well known that P^\perp is given by $P^\perp\omega = d(d^*d)^{-1}d^*\omega$ for $\omega \in H_1$ under various circumstances. This is the case here also when M is bounded and either Neumann or Dirichlet boundary conditions are used. But we will not need this expression.

Lemma 6.10 *For any $a \in [0, 1]$ the operator $P^\perp : H_a \rightarrow H_a$ is bounded. In particular, if $C(\cdot) : [0, T] \rightarrow H_a$ is continuous then the function*

$$\hat{C}(t) \equiv P^\perp C(t), \quad 0 \leq t \leq T \quad (6.15)$$

is also continuous into H_a

Proof. Writing $D = (1 - \Delta)^{1/2}$ as before, we have

$$\|D^a P^\perp\omega\|_2 \leq c_a \|D^a\omega\|_2, \quad a = 0, 1, \quad (6.16)$$

with $c_0 = 1$ because P^\perp is a projection on L^2 , and $c_1 \leq 2$ by Lemma 6.8. By complex interpolation (6.16) holds for all $a \in [0, 1]$ with $c_a \leq 2$. The assertion concerning \hat{C} now follows. ■

6.4 Integral representation of $g_\epsilon^{-1}dg_\epsilon$

To show that the functions $g(t, \cdot)$ constructed in Lemma 6.5 lie in the gauge group \mathcal{G}_{1+a} for each t we will need information about the spatial derivatives of g . The next proposition gives a representation of the spatial derivatives from which we will derive quantitative bounds in Sections 6.5 and 6.6. The simple representation of $g^{-1}dg$ which was used in [2] is inadequate for use in the estimates we will need in this paper. Instead we will use the representation in the next Proposition.

Proposition 6.11 (*Representation of $g_\epsilon^{-1}dg_\epsilon$*) Suppose that $C(\cdot)$ is the strong solution of the augmented equation (2.22) over $[0, T]$ constructed in Theorem 2.20 with initial value $C_0 = A_0$. Define g_ϵ as in (6.1) and define

$$h_\epsilon(t) = g_\epsilon(t)^{-1}dg_\epsilon(t), \quad 0 < t \leq T. \quad (6.17)$$

Define $\hat{C}(t)$ as in (6.15) and let

$$a_\epsilon(t, x) = \text{Ad}(g_\epsilon(t, x)^{-1}) \quad \text{for } 0 < t \leq T \text{ and } x \in M. \quad (6.18)$$

Then

$$h_\epsilon(t) = \left(\hat{C}(\epsilon) - a_\epsilon(t)\hat{C}(t) \right) + \int_\epsilon^t a_\epsilon(s)\chi(s)ds, \quad 0 < t \leq T, \quad (6.19)$$

where

$$\chi(s) = [\hat{C}(s), \phi(s)] - P^\perp \left(d_C^* B_C + [C, \phi] \right). \quad (6.20)$$

The proof depends on the following lemma.

Lemma 6.12 (*An identity*) Suppose that $C(\cdot)$ is a C^∞ solution to the augmented Yang-Mills heat equation (2.22) over some interval (a, b) . Let $g(t, x)$ be a smooth solution to the ODE

$$g'(t, x)g(t, x)^{-1} = d^*C(t, x) \quad (6.21)$$

for each $x \in M$ and $t \in (a, b)$. Define

$$a(t, x) = \text{Ad}(g(t, x)^{-1}) \quad \text{for } t \in (a, b) \text{ and } x \in M. \quad (6.22)$$

Let $\hat{C}(t) = P^\perp C(t)$ as in (6.15) and $\phi(t) = d^*C(t)$ as in (4.4). Then, for each (suppressed) $s \in (a, b)$, we have

$$d\phi = -\hat{C}' - P^\perp \left(d_C^* B_C + [C, \phi] \right). \quad (6.23)$$

where $\hat{C}'(s) = (d/ds)\hat{C}(s)$. Further,

$$\begin{aligned} (d/ds)(g(s)^{-1}dg(s)) &= - (d/ds) \left\{ a(s)\hat{C}(s) \right\} \\ &+ a(s) \left\{ [\hat{C}(s), \phi(s)] - P^\perp \left(d_C^* B_C + [C, \phi] \right) \right\}. \end{aligned} \quad (6.24)$$

Proof. The augmented heat equation asserts that $-C' = d_C^* B_C + d\phi + [C, \phi]$. Thus $d\phi = -\{C' + d_C^* B_C + [C, \phi]\}$. Since $d\phi$ is vertical and $P^\perp C'(s) = (d/ds)P^\perp C(s)$, we can apply the vertical projection P^\perp to this equation to find $d\phi = -\hat{C}' - P^\perp(d_C^* B_C + [C, \phi])$, which is (6.23).

For the derivation of (6.24) we need to use the following identity, which is valid for any continuous \mathfrak{k} valued 1-form ω on M .

$$a'(s)\omega = a(s)[\omega, \phi]. \quad (6.25)$$

This follows from the definitions (6.21) and (6.22) and the computation, at each (suppressed) point $x \in M$, $a'(s)\omega = (d/ds)(g(s)^{-1}\omega g(s)) = g^{-1}\omega(g'g^{-1})g - g^{-1}(g'g^{-1})\omega g = g^{-1}[\omega, \phi]g$.

From (6.25) and the product rule we have $(d/ds)(a\hat{C}) = a\hat{C}' + a'\hat{C} = a\hat{C}' + a[\hat{C}, \phi]$, so that

$$a\hat{C}' = (d/ds)(a\hat{C}) - a[\hat{C}, \phi]. \quad (6.26)$$

We will make use of the identity

$$(g(s)^{-1}dg(s))' = a(s)d\phi(s) \quad (6.27)$$

proved in [2, Eq (8.18)]. From (6.23), (6.26) and (6.27) it follows that

$$\begin{aligned} (g(s)^{-1}dg(s))' &= a(s) \left\{ -\hat{C}' - P^\perp \left(d_C^* B_C + [C, \phi] \right) \right\} \\ &= -(d/ds)(a\hat{C}) + a[\hat{C}, \phi] - aP^\perp \left(d_C^* B_C + [C, \phi] \right), \end{aligned}$$

which is (6.24). ■

Proof of Proposition 6.11. If we take g in Lemma 6.12 to be g_ϵ , defined in (6.1) then $h_\epsilon(t)$, defined in (6.17), satisfies $h_\epsilon(\epsilon) = 0$ in view of the initial condition in (6.1). Moreover $a_\epsilon(\epsilon)(x) =$ the identity operator on \mathfrak{k} for all $x \in M$. The identity (6.24) shows that

$$(d/ds)h_\epsilon(s) = -(d/ds)\left\{a_\epsilon(s)\hat{C}(s)\right\} + a_\epsilon(s)\chi(s). \quad (6.28)$$

We may integrate (6.28) to find

$$\begin{aligned} h_\epsilon(t) &= \int_\epsilon^t (d/ds)h_\epsilon(s)ds \\ &= \int_\epsilon^t \left(-(d/ds)\{a_\epsilon(s)\hat{C}(s)\} + a_\epsilon(s)\chi(s) \right) ds \\ &= -a_\epsilon(s)\hat{C}(s)\Big|_\epsilon^t + \int_\epsilon^t a_\epsilon(s)\chi(s)ds. \end{aligned}$$

This proves (6.19). ■

6.5 Estimates for $g_\epsilon^{-1}dg_\epsilon$

We need to make estimates of the integrand in the representation (6.19). Our estimates will be described in the following two theorems, which differ in the nature of their techniques of proof. The first depends entirely on the initial behavior estimates made in Section 4. The second depends on the multiplier bounds of Section 5.

Theorem 6.13 *Let $1/2 \leq a < 1$ and $0 < T < \infty$. Assume that M , A_0 and $C(\cdot)$ are as stated in Theorem 6.2. Define $\chi(s)$ as in (6.20). Let*

$$1/q_a = 1/2 - a/3. \quad (6.29)$$

Then

$$\int_0^T \|\chi(s)\|_{H_a} ds < \infty \quad \text{and} \quad (6.30)$$

$$\int_0^T \|\chi(s)\|_q ds < \infty \quad \text{for } 3 \leq q \leq q_a. \quad (6.31)$$

Theorem 6.14 *Under the same hypotheses as in Theorem 6.13 there holds*

$$\int_0^T \|a_\epsilon(s)\chi(s)\|_q ds \leq c_{21} \quad \forall \epsilon \in (0, T) \text{ and } 3 \leq q \leq q_a, \quad (6.32)$$

$$\sup_{\epsilon > 0, t > 0} \|h_\epsilon(t)\|_q < \infty \quad \text{for } 3 \leq q \leq q_a, \quad (6.33)$$

$$\sup_{\{\epsilon: 0 < \epsilon \leq t\}} \|h_\epsilon(t)\|_q \rightarrow 0 \quad \text{as } t \downarrow 0 \quad \text{for } 3 \leq q \leq q_a, \quad (6.34)$$

$$\sup_{0 < \delta \leq T} \int_0^T \|a_\delta(s)\chi(s)\|_{H_a} ds < \infty, \quad (6.35)$$

$$\sup_{\{\epsilon: 0 < \epsilon \leq t\}} \|h_\epsilon(t)\|_{H_a} \rightarrow 0 \quad \text{as } t \downarrow 0 \quad (6.36)$$

for some finite constant c_{21} depending only on $C(\cdot)$.

The order in (6.33) - (6.36) reflects the order in which the proof proceeds.

Corollary 6.15 *Suppose that M is as in the statement of Theorem 6.2. Let $C(\cdot)$ be a strong solution to the augmented Yang-Mills heat equation (2.22) over $[0, T]$ for some $T < \infty$. Define $\chi(s)$ as in (6.20). Then*

$$\int_{\epsilon_0}^T \|\chi(s)\|_3 ds < \infty \quad \text{for any } \epsilon_0 > 0 \quad \text{and} \quad (6.37)$$

$$\int_{\epsilon_0}^T \|\chi(s)\|_{H_1} ds < \infty \quad \text{for any } \epsilon_0 > 0. \quad (6.38)$$

Remark 6.16 (Strategy) A proof of the main inequality (6.30) will require a bound on $\|D^a \omega\|_2$ when $\omega = \chi(s)$ and $1/2 \leq a < 1$. This fractional derivative cannot be computed directly. Instead we will compute first order derivatives, $d\omega$ and $d^* \omega$ and make estimates of their L^p norms for “small” p , i.e. $p < 2$. Then we will implement the heuristic $\|D^a \omega\|_2 = \|D^{a-1} D \omega\|_2 \leq \|D^{a-1}\|_{L^p \rightarrow L^2} \|D \omega\|_p$, wherein the last inequality is a Sobolev inequality.

Lemma 6.17 (*Riesz avoidance*) *Let $0 \leq b \leq 1$. Define p in the interval $[6/5, 2]$ by*

$$\frac{1}{2} = \frac{1}{p} - \frac{(1-b)}{3}. \quad (6.39)$$

If ω is a 1-form in L^p with $d\omega \in L^p$ and $d^*\omega \in L^p$ then $\omega \in H_b$. There is a Sobolev constant $\kappa_{p,2}$ such that

$$\|\omega\|_{H_b} \leq \kappa_{p,2} \left(\|d\omega\|_p + \|d^*\omega\|_p \right) + \min(\kappa_{p,2}\|\omega\|_p, \|\omega\|_2). \quad (6.40)$$

In particular, if $\omega = P^\perp \mu$ for some 1-form μ then

$$\|\omega\|_{H_b} \leq \kappa_{p,2} \|d^*\mu\|_p + \|P^\perp \mu\|_2. \quad (6.41)$$

Proof. Let

$$D_j = (d^*d + dd^* + 1)^{1/2} \text{ on } j\text{-forms, } j = 0, 1, 2. \quad (6.42)$$

For any 1-form ω there holds

$$\|\omega\|_2^2 = \|dD_1^{-1}\omega\|_2^2 + \|d^*D_1^{-1}\omega\|_2^2 + \|D_1^{-1}\omega\|_2^2, \quad (6.43)$$

as follows from the computation

$$\begin{aligned} & (D_1^{-1}d^*dD_1^{-1}\omega, \omega) + (D_1^{-1}dd^*D_1^{-1}\omega, \omega) + (D_1^{-2}\omega, \omega) \\ &= (D_1^{-1}D_1^2D_1^{-1}\omega, \omega) = \|\omega\|_2^2. \end{aligned}$$

Hence

$$\begin{aligned} \|\omega\|_{H_b}^2 &= \|D_1^b\omega\|_2^2 \\ &= \|dD_1^{-1}D_1^b\omega\|_2^2 + \|d^*D_1^{-1}D_1^b\omega\|_2^2 + \|D_1^{-1}D_1^b\omega\|_2^2 \\ &= \|D_2^{b-1}d\omega\|_2^2 + \|D_0^{b-1}d^*\omega\|_2^2 + \|D_1^{b-1}\omega\|_2^2 \\ &\leq \|D_2^{b-1}\|_{p \rightarrow 2}^2 \|d\omega\|_p^2 + \|D_0^{b-1}\|_{p \rightarrow 2}^2 \|d^*\omega\|_p^2 + \|D_1^{b-1}\omega\|_2^2. \end{aligned}$$

Let $\kappa_{p,2} = \max(\|D_j^{a-1}\|_{p \rightarrow 2}^2, j = 0, 1, 2)$. Since D_1^{b-1} is both a contraction on L^2 and bounded from L^p to L^2 we have $\|D_1^{b-1}\omega\|_2 \leq \min(\kappa_{p,2}\|\omega\|_p, \|\omega\|_2)$. Therefore

$$\|\omega\|_{H_b}^2 \leq \kappa_{p,2}^2 (\|d\omega\|_p^2 + \|d^*\omega\|_p^2) + \min(\kappa_{p,2}\|\omega\|_p, \|\omega\|_2)^2,$$

which implies (6.40). In case $\omega = P^\perp \mu$ we have $d\omega = 0$ and $d^*\omega = d^*\mu$ by Lemma 6.8. The inequality (6.40) therefore implies (6.41) in this case. ■

6.5.1 Proof of Theorem 6.13

Lemma 6.18 (*Low p*) Let $3/2 \leq p \leq 3$ and $(1/r) = (1/p) - (1/6)$. Then

$$\|d^*\chi(s)\|_p + \|d\chi(s)\|_p \leq c_5 \|C(s)\|_{H_1} \left(\|d_C^* B_C(s)\|_r + \|d\phi(s)\|_r \right) \quad (6.44)$$

for a constant c_5 depending only on a Sobolev constant and the commutator bound c .

Proof. The first of the following three identities

$$d[\hat{C}(s), \phi(s)] = -[\hat{C}(s) \wedge d\phi(s)], \quad (6.45)$$

$$d^*[\hat{C}(s), \phi(s)] = [\hat{C}(s) \lrcorner d\phi(s)], \quad (6.46)$$

$$d^* \left(d_C^* B_C + [C, \phi] \right) = - \left[C \lrcorner \left(d_C^* B_C - d\phi \right) \right], \quad (6.47)$$

follows from the product rule: $d[\hat{C}(s), \phi(s)] = [d\hat{C}(s), \phi(s)] - [\hat{C}(s) \wedge d\phi(s)] = -[\hat{C}(s) \wedge d\phi(s)]$ because $d\hat{C}(s) = 0$ by (6.13). Since $[d^*\hat{C}, \phi] = [d^*C, \phi] = [\phi, \phi] = 0$, the second identity follows from the product rule $d^*[\hat{C}, \phi] = [d^*\hat{C}, \phi] + [C \lrcorner d\phi]$. In the third identity the second term is $d^*[C, \phi] = [C \lrcorner d\phi]$, while the Bianchi identity gives $d^*d_C^*B_C = d_C^*d_C^*B_C - [C \lrcorner d_C^*B_C] = -[C \lrcorner d_C^*B_C]$.

Since $d(P^\perp \omega) = 0$ for any 1-form $\omega \in L^2$, these three identities show that

$$\begin{aligned} \|d\chi(s)\|_p &= \|d[\hat{C}(s), \phi(s)] - dP^\perp \left(d_C^* B_C + [C, \phi] \right)\|_p \\ &= \|d[\hat{C}(s), \phi(s)]\|_p \\ &= \|[\hat{C}(s) \wedge d\phi(s)]\|_p \\ &\leq c \|\hat{C}(s)\|_6 \|d\phi(s)\|_r, \end{aligned} \quad (6.48)$$

and also

$$\begin{aligned} \|d^*\chi(s)\|_p &= \|d^*[\hat{C}(s), \phi(s)] - d^* \left(d_C^* B_C + [C, \phi] \right)\|_p \\ &= \|[\hat{C}(s) \lrcorner d\phi(s)] + \left[C \lrcorner \left(d_C^* B_C - d\phi \right) \right]\|_p \\ &\leq c \|\hat{C}(s)\|_6 \|d\phi(s)\|_r + c \|C(s)\|_6 \left\| \left(d_C^* B_C - d\phi \right) \right\|_r. \end{aligned} \quad (6.49)$$

Since P^\perp is a bounded operator from H_1 to H_1 (with bound at most $\sqrt{2}$) we have $\|\hat{C}(s)\|_6 \leq \kappa_6 \|\hat{C}(s)\|_{H_1} \leq 2\kappa_6 \|C(s)\|_{H_1}$ and also $\|C(s)\|_6 \leq \kappa_6 \|C(s)\|_{H_1}$.

Insert these bounds into (6.48) and (6.49) to find $\|d\chi(s)\|_p \leq 2c\kappa_6\|C(s)\|_{H_1}\|d\phi(s)\|_r$ and $\|d^*\chi(s)\|_p \leq 2c\kappa_6\|C(s)\|_{H_1}\|d\phi(s)\|_r + c\kappa_6\|C(s)\|_{H_1}\left(\|d_C^*B_C(s)\|_r + \|d\phi(s)\|_r\right)$. Add to arrive at (6.44) with $c_5 = 5c\kappa_6$. ■

Lemma 6.19 *Let $3/2 \leq p \leq 2$. Define $(1/r) = (1/p) - (1/6)$ and define b by (6.39). Then*

$$\|\chi(s)\|_{H_b} \leq \kappa_{p,2}c_5\|C(s)\|_{H_1}\left(\|d_C^*B_C(s)\|_r + \|d\phi(s)\|_r\right) \quad (6.50)$$

$$+ \|d_C^*B_C(s)\|_2 + 3c\kappa_6\|C(s)\|_{H_1}\|\phi(s)\|_3. \quad (6.51)$$

Proof. Choose $\omega = \chi(s)$ in (6.40) Then (6.40) and (6.44) show that

$$\begin{aligned} \|\chi(s)\|_{H_b} &\leq \kappa_{p,2}(\|d\chi(s)\|_p + \|d^*\chi(s)\|_p) + \|\chi(s)\|_2 \\ &\leq \kappa_{p,2}c_5\|C(s)\|_{H_1}\left(\|d_C^*B_C(s)\|_r + \|d\phi(s)\|_r\right) + \|\chi(s)\|_2. \end{aligned}$$

But

$$\begin{aligned} \|\chi(s)\|_2 &= \|\hat{C}(s), \phi(s)\| - P^\perp(d_C^*B_C(s) + [C(s), \phi(s)])\|_2 \\ &\leq \|\hat{C}(s), \phi(s)\|_2 + \|d_C^*B_C(s) + [C(s), \phi(s)]\|_2 \\ &\leq c(\|\hat{C}(s)\|_6 + \|C(s)\|_6)\|\phi(s)\|_3 + \|d_C^*B_C(s)\|_2 \\ &\leq 3\kappa_6\|C(s)\|_{H_1}\|\phi(s)\|_3 + \|d_C^*B_C(s)\|_2, \end{aligned}$$

wherein the two L^6 norms have been estimated in the last line just as in the proof of Lemma 6.18. ■

Lemma 6.20 *Let $1/2 \leq a \leq 1$. Define*

$$r_a^{-1} = (2/3) - (a/3). \quad (6.52)$$

Then, for any 1-form $u(s)$ over M there holds

$$\int_0^T s^a \|u(s)\|_{r_a}^2 ds \leq T^{a-(1/2)} \left(\int_0^T s^{1-a} \|u(s)\|_2^2 ds \right)^\alpha \left(\int_0^T s^{2-a} \|u(s)\|_6^2 ds \right)^\beta \quad (6.53)$$

where $\alpha = (3/2) - a$ and $\beta = a - (1/2)$.

Proof. The standard interpolation formula

$$\|\psi\|_r \leq \|\psi\|_2^\alpha \|\psi\|_6^\beta, \quad (6.54)$$

is valid for $2 \leq r \leq 6$, $\alpha = (3/r) - (1/2)$ and $\beta = (3/2) - (3/r)$. For $2 \leq r \leq 6$ both α and β are non-negative and $\alpha + \beta = 1$. Take $\psi = u(s)$ and observe that

$$s^a \|u(s)\|_r^2 \leq s^{a-\gamma} \left(s^{1-a} \|u(s)\|_2^2 \right)^\alpha \left(s^{2-a} \|u(s)\|_6^2 \right)^\beta, \quad (6.55)$$

where $\gamma = (1-a)\alpha + (2-a)\beta = 1-a+\beta = (5/2) - a - (3/r)$. In case $r = r_a$ we therefore have $a - \gamma = a - (1/2)$. Since $a - (1/2) \geq 0$ the first factor in (6.55) has a non-negative exponent. Integrate both sides of (6.55) over $(0, T]$, taking the maximum of the first factor out, and use Hölder's inequality to arrive at (6.53). ■

Proof of Theorem 6.13. Choosing $b = a$ in Lemma 6.19, we need only show that each of the four terms on the right hand side of (6.50) + (6.51) is integrable over $(0, T]$. Since $1/2 \leq a < 1$ we have $3/2 \leq p < 2$. The value of r determined in Lemma 6.19 for the value $b = a$ is given by $(1/r) = (1/p) - (1/6) = (1/2) + (1-a)/3 - (1/6) = (1/r_a)$. Thus we can apply Lemma 6.20.

For the integral of the first term in (6.50) we find

$$\begin{aligned} & \int_0^T \|C(s)\|_{H_1} \|d_C^* B_C(s)\|_{r_a} ds \\ & \leq \left(\int_0^T s^{-a} \|C(s)\|_{H_1}^2 ds \right)^{1/2} \left(\int_0^T s^a \|d_C^* B_C(s)\|_{r_a}^2 ds \right)^{1/2} \\ & \leq \left(\int_0^T s^{-a} \|C(s)\|_{H_1}^2 ds \right)^{1/2} \\ & \quad \left\{ T^{a-(1/2)} \left(\int_0^T s^{1-a} \|d_C^* B_C(s)\|_2^2 ds \right)^\alpha \left(\int_0^T s^{2-a} \|d_C^* B_C(s)\|_6^2 ds \right)^\beta \right\}^{1/2}. \end{aligned} \quad (6.56)$$

All three integrals are finite, the first by the assumption of finite strong a -action, the second by the inequality (4.53), and the third by the inequality (4.65).

The integral of the second term in (6.50) can be bounded the same way: One need only replace $\|d_C^* B_C(s)\|_{r_a}$ by $\|d\phi(s)\|_{r_a}$ in the inequalities (6.56).

The final step in the integrability argument holds again, in virtue of the inequalities (4.53) and (4.66).

Concerning the first term in line (6.51) we have, by (4.17) and (4.63), $\|d_C^* B_C(s)\|_2 \leq \|C'(s)\|_2 = o(s^{-1+(a/2)})$, which is integrable over $(0, T]$. The second term in line (6.51) is integrable by virtue of the inequalities $\left(\int_0^T \|C(s)\|_{H_1} \|\phi(s)\|_3 ds\right)^2 \leq \int_0^T s^{-a} \|C(s)\|_{H_1}^2 ds \int_0^T s^a \|\phi(s)\|_3^2 ds$, which is finite in view of (4.60), since $a \geq 1/2$. This proves (6.30).

Since $\|\chi(s)\|_b$ is dominated by $\|\chi(s)\|_a$ when $b \leq a$ it follows that $\int_0^T \|\chi(s)\|_{H_b} ds < \infty$ for $1/2 \leq b \leq a$. Thus if $3 \leq q \leq q_a$ and $q^{-1} = 2^{-1} - (b/3)$ then Sobolev gives (6.31). This completes the proof of Theorem 6.13. ■

6.5.2 Proof of Theorem 6.14

The following three lemmas prove Theorem 6.14.

Lemma 6.21 *There is a constant $c_{19} < \infty$, independent of ϵ and t , such that*

$$\|h_\epsilon(t)\|_q \leq c_{19} \quad \text{for } 0 < t \leq T, \quad 0 < \epsilon < T \quad \text{and } 3 \leq q \leq q_a. \quad (6.57)$$

Furthermore

$$\sup_{\{\epsilon: \epsilon \leq t\}} \|h_\epsilon(t)\|_q \rightarrow 0 \quad \text{as } t \downarrow 0 \quad \text{for } 3 \leq q \leq q_a. \quad (6.58)$$

Proof. Since the operators $a_\epsilon(s)$ are isometries in all L^p spaces, the representation (6.19) shows that

$$\begin{aligned} \|h_\epsilon(t)\|_q &\leq \|\hat{C}(\epsilon)\|_q + \|\hat{C}(t)\|_q + \left| \int_\epsilon^t \|\chi(s)\|_q ds \right| \\ &\leq 2 \sup_{0 < s \leq T} \|\hat{C}(s)\|_q + \int_0^T \|\chi(s)\|_q ds \quad \text{for } 0 < t \leq T. \end{aligned} \quad (6.59)$$

Lemma 6.10 shows that $\hat{C}(\cdot)$ is continuous on $[0, T]$ into H_a and therefore into H_b for all $b \in [1/2, a]$ and therefore into L^q for all $q \in [3, q_a]$. Hence, in view of (6.31), the right side of (6.59) is finite. This proves (6.57).

It might be useful to note that we have obtained a bound on $\|h_\epsilon(t)\|_q$ for all $q \in [3, q_a]$ by using $H_a \subset H_b$ if $a \geq b$ but not by using $L^{q_a} \subset L^q$ if $q_a > q$.

The latter would require M to be of finite volume, which we do not have when $M = \mathbb{R}^3$. The former holds because of the definition (2.10). We will need a bound on $\|h_\epsilon(t)\|_3$ in order to apply the multiplier bounds of Section 5.2.

Concerning the assertion (6.58), observe that the L^q norm of the integral term in the representation (6.19) is at most $\int_0^t \|\chi(s)\|_q ds$ for all $\epsilon \in (0, t]$, which goes to zero as $t \downarrow 0$ in view of (6.31). In regard to the integrated terms in (6.19), observe that for any norm we have

$$\begin{aligned} \|\hat{C}(\epsilon) - a_\epsilon(t)\hat{C}(t)\| &\leq \|\hat{C}(\epsilon) - \hat{C}(0)\| + \|(1 - a_\epsilon(t))\hat{C}(0)\| \\ &\quad + \|a_\epsilon(t)(\hat{C}(0) - \hat{C}(t))\|. \end{aligned} \quad (6.60)$$

If the norm is the L^q norm then, since $a_\epsilon(t)$ is isometric in this norm, the first and third terms on the right add to $\|\hat{C}(\epsilon) - \hat{C}(0)\|_q + \|\hat{C}(0) - \hat{C}(t)\|_q$, which goes to zero as $0 < \epsilon \leq t \downarrow 0$ because $\hat{C}(\cdot)$ is continuous into H_a , therefore into H_b , and therefore into L^q . The second term on the right side can (and must) be treated as a strong limit in the sense of Lemma 5.15. Since $\hat{C}(0)$ is fixed i.e. is independent of ϵ and t , Lemma 5.15 shows that this term will be small when $\|g_\epsilon(t) - I_V\|_2$ is small. The latter is assured by (6.11) with $p = 2$. This completes the proof of Lemma 6.21. ■

Lemma 6.22 (6.35) and (6.36) hold.

Proof. In view of (5.13) and (6.57) with $q = 3$ we have

$$\|a_\delta(s)\chi(s)\|_{H_a} \leq c_{20}\|\chi(s)\|_{H_a}, \quad 0 < \delta < T, \quad (6.61)$$

where $c_{20} = 1 + c_1 c_{19}$. Therefore

$$\left\| \int_0^T a_\delta(s)\chi(s)ds \right\|_{H_a} \leq c_{20} \int_0^T \|\chi(s)\|_{H_a} ds. \quad (6.62)$$

Since the right hand side is finite, by (6.30), and independent of δ the inequality (6.35) follows. Concerning the assertion (6.36) observe that, as in the case of the L^q norms, the integral in the representation (6.19) for $h_\epsilon(t)$ is at most $\int_0^t \|a_\epsilon(s)\chi(s)\|_{H_a} ds$, which is dominated by $c_{20} \int_0^t \|\chi(s)\|_{H_a} ds$ and which goes to zero as $t \downarrow 0$. To address the integrated terms in (6.19) take the norm in (6.60) to be the H_a norm. This time we need to base our estimates on the inequality (5.13), which gives

$$\begin{aligned} \|\hat{C}(\epsilon) - a_\epsilon(t)\hat{C}(t)\|_{H_a} &\leq \|\hat{C}(\epsilon) - \hat{C}(0)\|_{H_a} + \|(1 - a_\epsilon(t))\hat{C}(0)\|_{H_a} \\ &\quad + (1 + c_1\|g_\epsilon^{-1}dg_\epsilon(t)\|_3)\|\hat{C}(0) - \hat{C}(t)\|_{H_a}. \end{aligned} \quad (6.63)$$

The factor $\|g_\epsilon^{-1}dg_\epsilon(t)\|_3$ is bounded in ϵ and t by (6.33) (which is why (6.33) had to be proven first.) Since $\hat{C}(\cdot)$ is continuous into H_a the first and third terms go to zero as $t \downarrow 0$, uniformly for $0 < \epsilon \leq t$. As in the case of the q norms, the second term on the right side can (and must) be treated as a strong limit, but this time in the sense of Lemma 5.17, which requires that $\|g_\epsilon(t) - I_V\|_2$ go to zero, as assured by (6.11), and also that $\|g_\epsilon(t)^{-1}dg_\epsilon(t)\|_3$ go to zero, which is assured by (6.34). ■

The inequality (6.32) follows immediately from (6.31). All the other assertions of Theorem 6.14 have been proven in the lemmas.

Remark 6.23 One should contrast (6.30) with (6.38). When $A_0 \in H_a$ with $a < 1$ the allowed singularity in $\|\chi(s)\|_{H_1}$ as $s \downarrow 0$ will be too strong to ensure that $\int_0^T \|\chi(s)\|_{H_1} ds < \infty$. But (6.38) avoids the singularity at zero. Actually, third order initial behavior estimates, which are not in this paper, would show that $\|\chi(s)\|_{H_1}$ is bounded on $[\epsilon_0, T]$. But we only need the integrability asserted in Corollary 6.15.

Proof of Corollary 6.15. No assumption on the nature of the initial singularity of $C(\cdot)$ has been made in the statement of the corollary. In particular we are not assuming finite strong a -action. However the conclusion of the corollary concerns the behavior of $C(\cdot)$ only on the interval $[\epsilon_0, T]$. Since $C(\cdot)$ is a continuous function on $[\epsilon_0/2, T]$ into $H_1(M)$ it has finite strong a -action on the interval $[\epsilon_0/2, T]$ for any $a \in [1/2, 1)$. By shifting the origin over to $\epsilon_0/2$ we can, without loss of generality, assume that $C(\cdot)$ has finite strong a -action over $[0, T]$ for any $a \in [1/2, 1)$ that we choose. We will make this assumption and leave a unspecified for easy comparison with formulas that we have already developed. By doubling ϵ_0 we can continue to write the distance from the origin as ϵ_0 rather than $\epsilon_0/2$. It suffices to prove, therefore, that (6.37) and (6.38) hold under the assumption that $C(\cdot)$ has finite strong a -action over $[0, T]$.

(6.37) now follows from (6.31) because $q_a \geq 3$ for all $a \in [1/2, 1)$.

For the proof of (6.38) choose, in Lemma 6.19, $p = 2$, $b = 1$, and therefore $r = 3$. Then (6.50) and (6.51) assert that

$$\|\chi(s)\|_{H_1} \leq c_5 \|C(s)\|_{H_1} \left(\|d_C^* B_C(s)\|_3 + \|d\phi(s)\|_3 \right) \quad (6.64)$$

$$+ \|d_C^* B_C(s)\|_2 + 3c\kappa_6 \|C(s)\|_{H_1} \|\phi(s)\|_3. \quad (6.65)$$

The integral of line (6.65) over $[\epsilon_0, T]$ (in fact over $(0, T]$) has already been shown to be finite in the proof of Theorem 6.13. The first term in line (6.64) can be estimated as in (6.56) thus:

$$\begin{aligned} \int_{\epsilon_0}^T \|C(s)\|_{H_1} \|d_C^* B_C(s)\|_3 ds \\ \leq \left(\int_{\epsilon_0}^T s^{-a} \|C(s)\|_{H_1}^2 ds \right)^{1/2} \left(\int_{\epsilon_0}^T s^a \|d_C^* B_C(s)\|_3^2 ds \right)^{1/2}. \end{aligned}$$

The first factor is finite by finite strong a-action. It may be illuminating to note that the second integral would not necessarily be finite if it were extended down to $s = 0$ because, for any $a \in [1/2, 1)$, the power s^a would not be high enough to match with the use of the L^3 norm. (The distinction between strong and almost strong solutions can be traced back to this point.) But in our case, using $s^a = s^{2a-(3/2)} s^{(1-a)/2} s^{(2-a)/2}$ and the interpolation $\|f\|_3^2 \leq \|f\|_2 \|f\|_6$ we find

$$\begin{aligned} \int_{\epsilon_0}^T s^a \|d_C^* B_C(s)\|_3^2 ds \leq \left(\max_{\epsilon_0 \leq s \leq T} s^{2a-(3/2)} \right) \times \\ \left(\int_{\epsilon_0}^T s^{1-a} \|d_C^* B_C(s)\|_2^2 ds \right)^{1/2} \left(\int_{\epsilon_0}^T s^{2-a} \|d_C^* B_C(s)\|_6^2 ds \right)^{1/2}, \end{aligned}$$

which is finite by (4.53) and (4.65). The second term in line (6.64) can be estimated similarly. ■

6.6 Convergence of $g_\epsilon^{-1} dg_\epsilon$

Lemma 6.24 *Let $1/2 \leq a < 1$. Under the hypotheses of Theorem 6.2, there is a continuous function $h : [0, T] \rightarrow L^{q_a}$ such that $h(0) = 0$ and such that, for each number $t_1 \in (0, T]$, there holds*

$$\sup_{t_1 \leq t \leq T} \|h(t) - h_\epsilon(t)\|_{q_a} \rightarrow 0 \quad \text{as } \epsilon \downarrow 0. \quad (6.66)$$

Proof. The representation (6.19) for h_ϵ gives, for each (suppressed) $x \in M$,

$$h_\delta(t) - h_\epsilon(t) = \left(\hat{C}(\delta) - \hat{C}(\epsilon) \right) - \left(a_\delta(t) - a_\epsilon(t) \right) \hat{C}(t) \quad (6.67)$$

$$+ \int_\delta^\epsilon a_\delta(s) \chi(s) ds + \int_\epsilon^t (a_\delta(\epsilon) - 1) a_\epsilon(s) \chi(s) ds \quad \text{for } 0 < t \leq T. \quad (6.68)$$

Therefore, for $0 < \delta \leq \epsilon \leq t_1 \leq t \leq T$, we have

$$\|h_\delta(t) - h_\epsilon(t)\|_{q_a} \leq \|\hat{C}(\delta) - \hat{C}(\epsilon)\|_{q_a} + \|(a_\delta(\epsilon) - I_V)\{a_\epsilon(t)\hat{C}(t)\}\|_{q_a} \quad (6.69)$$

$$+ \int_\delta^\epsilon \|\chi(s)\|_{q_a} ds + \int_\epsilon^t \| |a_\delta(\epsilon) - 1|_{EndV} |\chi(s)|_\mathfrak{k} \|_{q_a} ds. \quad (6.70)$$

We need to show that each of these four terms go to zero uniformly for $t \in [t_1, T]$ as $0 < \delta \leq \epsilon \downarrow 0$.

Term # 1 goes to zero, uniformly for $t \in (0, T]$, as $0 < \delta \leq \epsilon \downarrow 0$ because $\hat{C}(\cdot)$ is continuous into H_a and therefore into L^{q_a} .

Term # 2 can be dominated for $0 < t_1 \leq t \leq T$ as follows. $\|a_\epsilon(t)\hat{C}(t)\|_6 \leq \sup_{t_1 \leq t \leq T} \|\hat{C}(t)\|_6 < \infty$ because \hat{C} is continuous into H_1 on $[t_1, T]$, hence into L^6 on this interval. Since $a < 1$ we have $q_a < 6$. But $\|a_\delta(\epsilon) - 1\|_p \rightarrow 0$ for all $p < \infty$ as a consequence of (6.8). Therefore Term # 2 converges to zero uniformly over $[t_1, T]$.

Term # 3 goes to zero, uniformly for $t \in (0, T]$, in view of (6.31).

Term # 4 goes to zero, uniformly for $t \in (0, T]$, because the integrand is dominated by the integrable function $2\|\chi(s)\|_{q_a}$ and goes to zero for each s because $\left(|a_\delta(\epsilon, x) - 1|_{End, \nu} |\chi(s)|_\mathfrak{k} \right)^{q_a} \leq |a_\delta(\epsilon, x) - 1|_{End, \nu}^{q_a} |\chi(s, x)|_\mathfrak{k}^{q_a}$, which goes to zero in measure and is dominated by $2^{q_a} |\chi(s, x)|^{q_a}$.

Hence there exists a function $h : (0, T] \rightarrow L^{q_a}$ to which the family $h_\epsilon(\cdot)$ converges for each $t \in (0, T]$ and uniformly on each interval $[t_1, T]$. Thus h is continuous from $(0, T]$ into L^{q_a} . Moreover $\|h(t)\|_{q_a} \leq \sup_{0 < \epsilon \leq t} \|h_\epsilon(t)\|_{q_a} \rightarrow 0$ as $t \downarrow 0$ by (6.34). Thus we may define $h(0) = 0$ to fulfill all the requirements of the lemma. ■

Lemma 6.25 *Let $1/2 \leq a < 1$. h is a continuous function on $[0, T]$ into H_a . Moreover, for any number $t_1 \in (0, T]$, there holds*

$$\sup_{t_1 \leq t \leq T} \|h(t) - h_\epsilon(t)\|_{H_a} \rightarrow 0 \quad \text{as } \epsilon \downarrow 0 \quad (6.71)$$

Proof. As in the proof of Lemma 6.24 we will show that the functions $h_\delta : [t_1, T] \rightarrow H_a$ form a uniformly Cauchy sequence. From the representation (6.67) we have, for $0 < \delta \leq \epsilon \leq t_1 \leq t \leq T$,

$$\begin{aligned} \|h_\delta(t) - h_\epsilon(t)\|_{H_a} &\leq \|\hat{C}(\delta) - \hat{C}(\epsilon)\|_{H_a} + \|(a_\delta(t) - a_\epsilon(t))\hat{C}(t)\|_{H_a} \\ &\quad + \int_\delta^\epsilon \|a_\delta(s)\chi(s)\|_{H_a} ds + \int_\epsilon^t \|(a_\delta(\epsilon) - 1)a_\epsilon(s)\chi(s)\|_{H_a} ds. \end{aligned} \quad (6.72)$$

We will show that each of the four terms on the right hand side go to zero as $0 < \delta \leq \epsilon \downarrow 0$.

Term # 1 goes to zero because $\hat{C}(\cdot)$ is a continuous function on $[0, T]$ into H_a .

Term # 2 can be dominated as follows. Choose $\delta_1 > 0$ such that $a + \delta_1 \leq 1$. Let $p_1 = 3/\delta_1$. Then, by (5.15) with $b = a$, we have

$$\begin{aligned} \|(a_\delta(t) - a_\epsilon(t))\hat{C}(t)\|_{H_a} &= \|(a_\delta(\epsilon) - 1)a_\epsilon(t)\hat{C}(t)\|_{H_a} \\ &\leq \left(\kappa_{\delta_1} \|g_\delta(\epsilon) - I_V\|_{p_1} + c_1 \|g_\delta(\epsilon)^{-1} dg_\delta(\epsilon)\|_3 \right) \|a_\epsilon(t)\hat{C}(t)\|_{H_{a+\delta_1}}. \end{aligned} \quad (6.73)$$

The two terms in parentheses go to zero by (6.8) (with $p = p_1$) and by (6.34) (with $q = 3$). Now $C(\cdot) : [t_1, T] \rightarrow H_1$ is continuous because $C(\cdot)$ lies in the path space \mathcal{P}_T^a . See Notation 3.2. By Lemma 6.10 $\hat{C}(t)$ is also continuous into H_1 and therefore also continuous into $H_{a+\delta_1}$. Hence

$$\sup_{t_1 \leq t \leq T} \|\hat{C}(t)\|_{H_{a+\delta_1}} < \infty \quad (6.74)$$

if $a + \delta_1 \leq 1$. The inequality

$$\|a_\epsilon(t)\hat{C}(t)\|_{H_{a+\delta_1}} \leq (1 + c_1 \|h_\epsilon(t)\|_3) \|\hat{C}(t)\|_{H_{a+\delta_1}} \quad (6.75)$$

follows from (5.13). In view of (6.33) with $q = 3$, the last factor in (6.73) is therefore bounded over $[t_1, T]$. Hence Term # 2 goes to zero uniformly for $t \in [t_1, T]$ as $0 < \delta \leq \epsilon \downarrow 0$.

Term # 3 is dominated by $c_{20} \int_\delta^\epsilon \|\chi(s)\|_{H_a} ds$ by (6.61). In view of (6.30) this term also goes to zero, and uniformly for $t \in (0, T]$.

Term # 4 can be estimated as follows. Suppose that $\alpha > 0$ is small. Choose $\epsilon_0 \in (0, T]$ so small that

$$2c_{20} \int_0^{\epsilon_0} \|\chi(s)\|_{H_a} ds < \alpha. \quad (6.76)$$

(6.30) assures that such an ϵ_0 exists. Then, for $0 < \delta \leq \epsilon \leq \epsilon_0$ and for all $t \in [\epsilon, T]$, we find, with the help of (6.61),

$$\begin{aligned}
& \left\| \int_{\epsilon}^t (a_{\delta}(s) - a_{\epsilon}(s)) \chi(s) ds \right\|_{H_a} \\
& \leq \int_{\epsilon}^{\epsilon_0} \|(a_{\delta}(s) - a_{\epsilon}(s)) \chi(s)\|_{H_a} ds + \int_{\epsilon_0}^T \|(a_{\delta}(s) - a_{\epsilon}(s)) \chi(s)\|_{H_a} ds \\
& \leq 2c_{20} \int_{\epsilon}^{\epsilon_0} \|\chi(s)\|_{H_a} ds + \int_{\epsilon_0}^T \|(a_{\delta}(s) - a_{\epsilon}(s)) \chi(s)\|_{H_a} ds \\
& < \alpha + \int_{\epsilon_0}^T \|(a_{\delta}(s) - a_{\epsilon}(s)) \chi(s)\|_{H_a} ds
\end{aligned} \tag{6.77}$$

for all $t \in [\epsilon, T]$ and for all $0 < \delta \leq \epsilon \leq \epsilon_0$. It remains to show that the integral in line (6.77) is small for small ϵ .

Choose again $\delta_1 > 0$ such that $a + \delta_1 \leq 1$ and let $p_1 = 3/\delta_1$. We can apply (5.15) with $b = a$ and $g = g_{\delta}(\epsilon)$ to find

$$\begin{aligned}
& \int_{\epsilon_0}^T \|(a_{\delta}(s) - a_{\epsilon}(s)) \chi(s)\|_{H_a} ds = \int_{\epsilon_0}^T \|(a_{\delta}(\epsilon) - 1) a_{\epsilon}(s) \chi(s)\|_{H_a} ds \\
& \leq \left(\kappa_{\delta_1} \|Ad g_{\delta}(\epsilon) - 1\|_{p_1} + c_1 \|h_{\delta}(\epsilon)\|_3 \right) \int_{\epsilon_0}^T \|a_{\delta}(s) \chi(s)\|_{H_{a+\delta_1}} ds \\
& \leq \left(\kappa_{\delta_1} \|Ad g_{\delta}(\epsilon) - 1\|_{p_1} + c_1 \|h_{\delta}(\epsilon)\|_3 \right) c_{20} \int_{\epsilon_0}^T \|\chi(s)\|_{H_{a+\delta_1}} ds.
\end{aligned} \tag{6.78}$$

In the last line we have used (6.61). The integral in the last line is finite by (??) with $b = a + \delta_1$. For any $p_1 < \infty$ the expression in large parentheses in (6.78) goes to zero as $0 < \delta \leq \epsilon \downarrow 0$ by (6.8) and (6.34).

Thus Term #4 goes to zero uniformly for t in any interval $[t_1, T]$ when $t_1 > 0$.

This concludes the proof that the functions $h_{\epsilon}(\cdot)$ are uniformly Cauchy in H_a norm over each interval $[t_1, T]$. The family of functions therefore converges to a continuous function $h(\cdot)$ into H_a over $(0, T]$, and since the H_a norm dominates the L^{q_a} norm the function h is the same as the one in Lemma 6.24, over $(0, T]$. Since $\|h(t)\|_{H_a} \leq \sup_{\epsilon \leq t} \|h_{\epsilon}(t)\|_{H_a}$, which goes to zero by (6.36) as $t \downarrow 0$, it follows that $\|h(t)\|_{H_a} \rightarrow 0$ as $t \downarrow 0$. Since $h(0) = 0$ by Lemma 6.24, this concludes the proof of Lemma 6.25. ■

6.7 Smooth ratios

Lemma 6.26 (*Smooth ratio*) Let $0 < \tau < T$. Define a function $k : (0, T] \times M \rightarrow K$ by

$$g(t) = k(t)g(\tau) \quad \text{on } (0, T] \times M, \quad (6.79)$$

where $g(t)$ is the gauge function constructed in Lemma 6.5. Then $k \in C^\infty((0, T) \times M; K)$ and, for each (suppressed) point $x \in M$, is the solution to

$$k'(t)k(t)^{-1} = d^*C(t), \quad 0 < t < T, \quad k(\tau) = I_{\mathcal{V}}. \quad (6.80)$$

$k(\cdot)$ satisfies the boundary conditions

$$(k(t)^{-1}dk(t))_{\text{norm}} = 0 \quad \text{for } 0 < t < T \quad \text{in case } (N) \quad (6.81)$$

$$(k(t)^{-1}dk(t))_{\text{tan}} = 0 \quad \text{for } 0 < t < T \quad \text{in case } (D). \quad (6.82)$$

Proof. For each point $x \in M$ let $u(t, x)$ be the unique solution to $u'(t, x)u(t, x)^{-1} = d^*C(t)$ on $(0, T)$ for which $u(\tau, x) = I_{\mathcal{V}}$. Then $u(t, x)$ lies in K for all $t \in (0, T]$ and all $x \in M$. If $\epsilon < \tau$ then $g_\epsilon(t) = u(t)g_\epsilon(\tau)$ for $\epsilon \leq t < T$ because both sides satisfy the ODE in (6.1) and agree at $t = \tau$. For fixed $t > 0$, $g_\epsilon(t)$ and $g_\epsilon(\tau)$ converge to $g(t)$ and $g(\tau)$, respectively, in $L^p(M; \text{End } \mathcal{V})$ by Lemma 6.5, as $\epsilon \downarrow 0$. Hence $g(t) = u(t)g(\tau)$ for $0 < t < T$. Therefore $k = u$. Since $C(\cdot) \in C^\infty((0, T) \times M)$ so is u and hence k .

The boundary conditions (6.81) and (6.82) follow from the boundary conditions (2.23), respectively (2.24) for $C(\cdot)$ by the same argument given in [2, Lemma 8.7]. ■

Lemma 6.27 Suppose that M is as in the statement of Theorem 6.2. Let $C(\cdot)$ be a strong solution to the augmented Yang-Mills heat equation (2.22) over $[0, T]$ for some $T < \infty$. Let $\tau > 0$ and let $k(\cdot)$ be the solution to the initial value problem (6.80). Then for $0 < \epsilon_0 \leq \tau$ there holds

$$\sup_{\epsilon_0 \leq t \leq T} \|k(t)^{-1}dk(t)\|_3 < \infty, \quad (6.83)$$

$$\sup_{\epsilon_0 \leq t \leq T} \|k(t)^{-1}dk(t)\|_{H_1} < \infty. \quad (6.84)$$

Proof. Since we are only concerned with the behavior of $k(t)$ for $t \geq \epsilon_0$ we can assume without loss of generality, by the argument in the proof of Corollary 6.15, that $C(\cdot) \in \mathcal{P}_T^a$ for any $a \in [1/2, 1)$. In Lemma 6.12 choose

for the function g the function k defined in (6.80). Since $dk(\tau) = 0$, we learn from (6.24) that

$$k(t)^{-1}dk(t) = \left(a(s)\hat{C}(s)\right)\Big|_t^\tau + \int_\tau^t a(s)\chi(s)ds, \quad t > 0, \quad (6.85)$$

where $a(s) = Ad\ k(s)^{-1}$ and $\chi(s)$ is again given by (6.20). Then

$$\begin{aligned} \|k(t)^{-1}dk(t)\|_3 &\leq \left\|\left(a(s)\hat{C}(s)\right)\Big|_t^\tau\right\|_3 \pm \int_\tau^t \|\chi(s)\|_3 ds \\ &\leq \|\hat{C}(\tau)\|_3 + \|\hat{C}(t)\|_3 + \int_{\epsilon_0}^T \|\chi(s)\|_3 ds \end{aligned}$$

The integral is finite by (6.37). $\hat{C}(\cdot)$ is a continuous function into H_a by Lemma 6.10 and therefore into $H_{1/2}$ and therefore into $L^3(M)$. Hence $\|\hat{C}(t)\|_3$ is bounded on $[\epsilon_0, T]$. This proves (6.83).

To prove (6.84) we will use the representation (6.85) again. In view of (5.13) we have, for $\epsilon_0 \leq t \leq T$,

$$\|k(t)^{-1}dk(t)\|_{H_1} \leq \left(1 + c_1\gamma_3\right)\left(\|\hat{C}(\tau)\|_{H_1} + \|\hat{C}(t)\|_{H_1} + \int_{\epsilon_0}^T \|\chi(s)\|_{H_1} ds\right)$$

where γ_3 denotes the left side of (6.83). $C(\cdot)$, and therefore $\hat{C}(\cdot)$, are continuous functions on $[\epsilon_0, T]$ into H_1 because $C(\cdot)$ lies in \mathcal{P}_T^a . The lemma now follows from (6.38). ■

6.8 Proof of Theorem 6.2

Most of the steps in the proof of Theorem 6.2 have been carried out in the preceding subsections. In Section 6.2 we showed that the functions $g_\epsilon : (0, T] \times M \rightarrow K \subset End\mathcal{V}$ converge as functions of t into $L^p(M; End\mathcal{V})$ and in fact uniformly for $t \in (0, T]$. To prove convergence in the sense of the metric groups \mathcal{G}_{1+a} one must show that the logarithmic derivatives $h_\epsilon(t) \equiv g_\epsilon(t)^{-1}dg_\epsilon(t)$ converge in H_a . It was first shown that the functions $h_\epsilon(\cdot)$ converge in L^{q_a} , in Lemma 6.24, and then shown, in Lemma 6.25, that they also converge in H_a , with both convergences uniform for $t \in [t_1, T]$ for each $t_1 \in (0, T]$. Therefore the functions $g_\epsilon(t)$ converge uniformly over $[t_1, T]$ to $g(t)$ in the sense of the metric group \mathcal{G}_{1+a} . $h(\cdot)$ and $g(\cdot)$ are therefore continuous on $(0, T]$ into H_a and \mathcal{G}_{1+a} respectively.

The limit function $h(t)$ converges to zero in H_a as $t \downarrow 0$ by virtue of (6.36). The limit function $g(t)$ therefore converges to the identity operator on $L^2(M; \mathcal{V})$ in the sense of the G_{1+a} topology as $t \downarrow 0$. h and g are therefore continuous into H_a and \mathcal{G}_{1+a} , respectively, over $[0, T]$. This proves Theorem 6.2, Parts b), c) and d).

The smoothness of ratios asserted in Part e) of Theorem 6.2 is proved in Lemma 6.26 because the function $g(t)$ constructed in Lemma 6.5 is the function g defined in Theorem 6.2 as a limit in the group \mathcal{G}_{1+a} . Since $k(t) = g(t)g(\tau)^{-1}$ and $g(t) \rightarrow I_{\mathcal{V}}$ in the sense of the metric group \mathcal{G}_{1+a} it follows that $k(t) \rightarrow g(\tau)^{-1}$ in this sense also. This concludes the proof of Theorem 6.2.

7 Recovery of A from C

In this section M will be assumed to be all of \mathbb{R}^3 or the closure of a bounded, convex, open subset of \mathbb{R}^3 with smooth boundary.

Theorem 7.1 (*Recovery of A from C*) *Let $1/2 \leq a < 1$. Let $M = \mathbb{R}^3$ or be the closure of a bounded, convex, open set in \mathbb{R}^3 with smooth boundary. Suppose that $A_0 \in H_a(M)$ and that $C(\cdot)$ is a strong solution to the augmented equation (2.22) with $C(0) = A_0$ and having finite strong a -action over $[0, T]$. Then there exists a continuous function*

$$g : [0, T] \rightarrow \mathcal{G}_{1+a} \quad (7.1)$$

such that $g(0) = I_{\mathcal{V}}$ and such that the gauge transform $A(\cdot)$, defined by

$$A(t) = C(t)^{g(t)}, \quad 0 \leq t \leq T, \quad (7.2)$$

is an almost strong solution to the Yang-Mills heat equation over $(0, T]$, whose curvature satisfies the boundary condition (2.15) resp. (2.16). The map

$$A(\cdot) : [0, T] \rightarrow H_a \quad (7.3)$$

is continuous. In particular $A(t)$ converges in H_a norm to A_0 as $t \downarrow 0$.

If $0 < \tau < T$ and $g_0 \equiv g(\tau)^{-1}$ then the function $t \mapsto A(t)^{g_0}$ is a strong solution to the Yang-Mills heat equation satisfying the boundary condition (2.15) resp (2.16) as well as the boundary condition (2.17) resp. (2.18). $A^{g_0}(\cdot)$ lies in $C^\infty((0, T) \times M; \Lambda^1 \otimes \mathfrak{k})$. The map

$$A^{g_0}(\cdot) : [0, T] \rightarrow H_a \quad (7.4)$$

is continuous. In particular $A(t)^{g_0}$ converges in H_a norm to $A_0^{g_0}$ as $t \downarrow 0$.

$A(\cdot)$ and $A^{g_0}(\cdot)$ have finite a -action:

$$\int_0^T s^{-a} \|B(s)\|_2^2 ds < \infty. \quad (7.5)$$

In case $a = 1/2$ and $\|A_0\|_{H_{1/2}}$ is sufficiently small then $C(\cdot)$ has finite strong $(1/2)$ -action and all the preceding conclusions hold.

If $a > 1/2$ then the solution $C(\cdot)$ to the augmented Yang-Mills equation automatically has finite a -action, as was proven in Theorem 3.18. If $a = 1/2$ and $C(\cdot)$ does not have finite $(1/2)$ -action then there is a weaker version of Theorem 7.1 that holds.

Theorem 7.2 (*Recovery in case of infinite action*) Let $M = \mathbb{R}^3$ or be the closure of a bounded, convex, open set in \mathbb{R}^3 with smooth boundary. Suppose that $A_0 \in H_{1/2}$ and that $C(\cdot)$ is a strong solution to the augmented equation (2.22) with not necessarily finite strong $1/2$ -action. Then there exists a continuous function

$$g : [0, T] \rightarrow \mathcal{G}_{1,2} \quad (7.6)$$

such that $g(0) = I_V$ and such that the function $A(t)$, defined by (7.2), is an almost strong solution to the Yang-Mills heat equation over $(0, T]$. Its curvature satisfies the boundary condition (2.15) resp. (2.16). If $g_0 = g(\tau)^{-1}$ as in Theorem 2.22 then $A(t)^{g_0}$ is a strong solution to the Yang-Mills heat equation satisfying (2.15) resp (2.16) as well as (2.17) resp. (2.18). $A(t)$ converges to A_0 in $L^2(M)$ and $A(t)^{g_0}$ converges to $A_0^{g_0}$ in $L^2(M)$.

Actually, the function $g(\cdot)$ on $[0, T]$ that we will construct in the proof of Theorem 7.2 will be a continuous function into the gauge group $\mathcal{G}_{1,q}$ for any $q \in [2, 3)$. See Remark 7.5 for this marginal improvement.

Remark 7.3 Theorems 7.1 and 7.2 prove and extend Theorem 2.22, to all $a \in [1/2, 1)$ and to infinite action. They will be used in the next subsection to prove the existence portions of the two main theorems, Theorems 2.10 and 2.11. The uniqueness assertions of these theorems will be proven in Section 7.3 after establishing apriori initial behavior properties of solutions $A(\cdot)$ to (2.5).

7.1 Construction of A

In this section we will prove Theorem 7.1 and its special case Theorem 2.22. We will also prove Theorem 7.2.

Proof of Theorem 7.1. Denote by $g(t)$ the function constructed from $C(\cdot)$ in Theorem 6.2. In view of (6.79) we may write the function $A(\cdot)$ defined in (2.27) as

$$A(t) = C(t)^{g(t)} = (C(t)^{k(t)})^{g(\tau)}. \quad (7.7)$$

Let

$$\hat{A}(t) = C(t)^{k(t)}, \quad 0 < t < T. \quad (7.8)$$

Then

$$A(t) = \hat{A}(t)^{g(\tau)}. \quad (7.9)$$

\hat{A} is a smooth function on $(0, T) \times M$ because C and k are smooth. Moreover (6.80) and (2.22) imply that \hat{A} is a (smooth) solution to the Yang-Mills heat equation, (2.5) over $(0, T)$. See [2, Lemma 8.6] for a proof. This is the ZDS mechanism for constructing a solution of (2.5) from a solution of (2.22). In accordance with [2, Lemma 8.6], the curvature and time derivative of \hat{A} and A can be expressed in terms of $C(\cdot)$ as

$$\hat{B}(t) = k(t)^{-1} B_C(t) k(t), \quad \hat{A}'(t) = k(t)^{-1} \left(d_C^* B_C(t) \right) k(t) \quad (7.10)$$

$$B(t) = g(t)^{-1} B_C(t) g(t) = g(\tau)^{-1} \hat{B}(t) g(\tau) \quad (7.11)$$

$$A'(t) = g(t)^{-1} \left(d_C^* B_C(t) \right) g(t) = g(\tau)^{-1} \hat{A}'(t) g(\tau) \quad (7.12)$$

Since $A(\cdot)$ is the gauge transform of $\hat{A}(\cdot)$ by a fixed gauge function $g(\tau)$, it is also a solution to the Yang-Mills heat equation, at least informally. We need to show that $\hat{A}(\cdot)$ is actually a strong solution and that $A(\cdot)$ is actually an almost strong solution.

By Corollary 4.18 $\|B_C(t)\|_\infty$ is bounded on $[\epsilon, T]$ for any $\epsilon > 0$. Secondly, $B_C(t) \in H_1(M)$ because $C(\cdot)$ is, by assumption, a strong solution to the augmented Yang-Mills heat equation, (2.22). Thirdly, $k(t)^{-1} dk(t) \in L^2(M)$ and $g(t)^{-1} dg(t) \in L^2(M)$. By the product rule, it follows from these three facts and the representations (7.10) and (7.11) that both $\hat{B}(t)$ and $B(t)$ are in $W_1(M)$ for each $t > 0$. Boundary conditions will be discussed below. Now $d_C^* B_C(t) \in L^2(M)$ by (4.17) and (4.63). Therefore (7.10) shows that $\hat{A}'(t) \in L^2(M)$ for $t > 0$. Either of the two representations in (7.12) shows

that $A'(t) \in L^2(M)$ for $t > 0$ also. Since $g(\cdot)$ and $k(\cdot)$ are both continuous into $\mathcal{G}_{1+a} \subset \mathcal{G}_{1,2}$ it is routine to show that \hat{A}' and A' are both continuous into $L^2(M)$. Therefore $\hat{A}(\cdot)$ and $A(\cdot)$ are both almost strong solutions to (2.5).

There is a distinction now between $k(t)$ and $g(t)$. In accordance with (6.2) we know that $g(t)^{-1}dg(t)$ lies in $H_a(M)$, but since $a < 1$ we cannot conclude that $A(t)$, which is $g^{-1}Cg + g^{-1}dg$, lies in $W_1(M)$. That is, (2.2) may fail and $A(\cdot)$ may therefore not be a strong solution. On the other hand Lemma 6.27 shows that $k(t)^{-1}dk(t) \in H_1(M)$ for all $t > 0$. Thus to show that $\hat{A}(t) \in H_1(M)$ it remains only to show that $k(t)^{-1}C(t)k(t) \in H_1(M)$ for each $t > 0$. But, in view of (5.13), this follows from the fact that $C(t) \in H_1(M)$ and $k(t)^{-1}dk(t) \in L^3(M)$, which has been shown in (6.83). Therefore $\hat{A}(\cdot)$ is a strong solution.

The boundary conditions (2.15) - (2.18) for \hat{A} and its curvature \hat{B} follow from (6.81) and (2.23), respectively (6.82) and (2.24), by the same argument as in [2, Corollary 8.8]. Since $B(t) = g(\tau)^{-1}\hat{B}(t)g(\tau)$ the boundary conditions (2.15), respectively (2.16), hold for $B(t)$ also. But it is well to note at this point that in the important case when $a = 1/2$ we do not know that $g(t)^{-1}dg(t)$ satisfies any particular boundary conditions and may not even have well defined boundary values because we know only that it lies in $H_{1/2}$. (See Theorem 6.2, Part d.) We therefore cannot assert an analog of (2.17) or (2.18) for A itself. By (7.9) we see that $A(t)^{g(\tau)^{-1}} = \hat{A}(t)$, from which it follows that $A^{g_0}(\cdot)$ lies in $C^\infty((0, T) \times M; \Lambda^1 \otimes \mathfrak{k})$, as asserted in the theorem.

Concerning the continuity of the map

$$A(\cdot) : [0, T] \rightarrow H_a, \quad (7.13)$$

observe that $A(t) = g(t)^{-1}C(t)g(t) + g(t)^{-1}dg(t)$, wherein the second term is a continuous function on $[0, T]$ into H_a by virtue of Theorem 6.2, Part d). The first term lies in H_a for every $t \in [0, T]$ by virtue of the inequality (see (5.13)) $\|g(t)^{-1}C(t)g(t)\|_{H_a} \leq (1 + c_1\|g(t)^{-1}dg(t)\|_3)\|C(t)\|_{H_a}$. For the continuity of the first term at a point $s \in [0, T]$ we have

$$\begin{aligned} & \| (Ad\ g(t)^{-1})C(t) - (Ad\ g(s)^{-1})C(s) \|_{H_a} \leq \| (Ad\ g(t)^{-1})(C(t) - C(s)) \|_{H_a} \\ & \quad + \| \{ (Ad\ g(t)^{-1}) - (Ad\ g(s)^{-1}) \} C(s) \|_{H_a} \\ & \leq (1 + c_1\|g(t)^{-1}dg(t)\|_3)\|C(t) - C(s)\|_{H_a} \end{aligned} \quad (7.14)$$

$$+ \| \{ (Ad\ g(t)^{-1}) - (Ad\ g(s)^{-1}) \} C(s) \|_{H_a}. \quad (7.15)$$

The first factor in line (7.14) is bounded because $t \mapsto g(t)^{-1}dg(t)$ is a continuous function into H_a and therefore into $H_{1/2}$ and therefore into $L^3(M)$.

Hence, since $C(\cdot)$ is a continuous function into H_a , Line (7.14) goes to zero as $t \rightarrow s$. In Line (7.15) s is fixed and we can therefore use the strong continuity of the representation $\mathcal{G}_{1+a} \ni g \mapsto Ad\ g|H_a$, as in Corollary 5.5, to conclude that Line (7.15) also goes to zero as $t \rightarrow s$. Thus $A(\cdot)$ is a continuous function on $[0, T]$ into H_a and in particular $A(t)$ converges to A_0 in H_a norm as $t \downarrow 0$ (and not just in L^2). Finally, in view of (5.13), we have $\|A(t)^{g_0} - A(s)^{g_0}\|_{H_a} = \|(Ad\ g_0^{-1})(A(t) - A(s))\|_{H_a} \leq (1 + c_1\|g_0^{-1}dg_0\|_3)\|A(t) - A(s)\|_{H_a} \rightarrow 0$ as $t \rightarrow s$. Herein we have used the fact that $g_0 = g(\tau)^{-1} \in \mathcal{G}_{1+a} \subset \mathcal{G}_{3/2} \subset \mathcal{G}_{1,3}$. Thus $A(\cdot)^{g_0}$ is also a continuous function on $[0, T]$ into H_a and in particular converges to its initial value $A_0^{g_0}$ in H_a norm.

That $A(\cdot)$ and $A(\cdot)^{g_0}$ have finite a -action when $C(\cdot)$ has finite strong a -action follows from (4.43) since gauge invariance shows that

$$\int_0^T s^{-a} \|B(s)\|_2^2 ds = \int_0^T s^{-a} \|B_C(s)\|_2^2 ds < \infty.$$

In case $a = 1/2$ and $\|A_0\|_{H_{1/2}}$ is sufficiently small then Theorem 3.18 shows that the solution $C(\cdot)$ to the augmented equation with initial data A_0 has finite strong action. Therefore all of the preceding assertions in Theorem 7.1 hold. This completes the proof of Theorem 7.1 and its special case Theorem 2.22. ■

Remark 7.4 It was pointed out in the introduction that if $A_0 := u^{-1}du$ is a pure gauge in $H_{1/2}(M)$ then the solution to the Yang-Mills heat equation is given by $A(t) := A_0$, which will never be in $H_1(M)$ if $A_0 \notin H_1(M)$. This is a simple example of an almost strong solution which is not a strong solution.

Proof of Theorem 7.2. The proof relies on the weaker estimates for infinite action proved in Section 4.6. We are going to use the simple expression (6.27) for $(d/ds)(g^{-1}dg)$ rather than the more complicated expression (6.24) because the latter does not offer an advantage now. Thus we have

$$h_\epsilon(t) = \int_\epsilon^t a(s) d\phi(s) ds \tag{7.16}$$

and therefore, for $1 \leq q \leq \infty$ we have

$$\begin{aligned} \|h_\epsilon(t)\|_q &\leq \left| \int_\epsilon^t \|a(s) d\phi(s)\|_q ds \right| \\ &= \left| \int_\epsilon^t \|d\phi(s)\|_q ds \right|. \end{aligned} \tag{7.17}$$

The case of immediate interest for us is $q = 2$. For $\delta > 0$ we have

$$\int_0^T \|d\phi(s)\|_2 ds \leq \left(\int_0^T s^{-(1/2)-\delta} ds \right)^{1/2} \left(\int_0^T s^{(1/2)+\delta} \|d\phi(s)\|_2^2 ds \right)^{1/2} < \infty$$

by (4.83) if $0 < \delta < 1/2$. Therefore $\|h_\epsilon(t)\|_2$ remains bounded on $(0, T]$ as $\epsilon \downarrow 0$. The standard machinery for differences, already used in Section 6.6, now shows that the functions $h_\epsilon(t)$ converge uniformly on $(0, T]$, as functions into $L^2(M; \Lambda^1 \otimes \mathfrak{k})$, to a continuous function h on $(0, T]$ into $L^2(M; \Lambda^1 \otimes \mathfrak{k})$ with limit $\lim_{t \downarrow 0} h(t) = 0$. Defining $h(0) = 0$ extends h to a continuous function on $[0, T]$ into $L^2(M; \Lambda^1 \otimes \mathfrak{k})$. The same arguments used in the proof of Theorem 6.2 now show that there is a continuous function $g : [0, T] \rightarrow \mathcal{G}_{1,2}$ such that $g(0, x) = I_{\mathcal{V}}$, and to which the functions $g_\epsilon(t, x)$ converge, uniformly over $(0, T]$, as functions into the metric group $\mathcal{G}_{1,2}$.

We need to show now that the gauge transform $A := C^g$ is an almost strong solution of the Yang-Mills heat equation (2.5) and that A^{g_0} is a strong solution. As in the case of finite strong a-action we have $\|B_C(t)\|_\infty < \infty$ for each $t > 0$ by Corollary 4.18 and $B_C(t) \in H_1(M)$. The proof that $g(t)^{-1}B_C(t)g(t) \in W_1(M)$ is therefore the same as for the case of finite strong a-action because that proof made use only of these two properties of $B_C(t)$ and the fact that $g(t) \in \mathcal{G}_{1,2}$. The same argument applies to $\hat{B}(t)$ in view of (7.10). Each lies in $H_1(M)$ by the same argument as in the strong a-action case. As in the case of strong a-action, $A'(t)$ and $\hat{A}'(t)$ both lie in $L^2(M)$.

Just as in the case of finite strong a-action, $A(t)$ can fail to lie in $W_1(M)$, whereas $\hat{A}(t)$ does lie in $H_1(M)$, the latter by virtue of Lemma 6.27 (for size) again and [2, Corollary 8.8] (for boundary conditions).

Since $g(\cdot)$ and $k(\cdot)$ are both continuous functions on $[0, T]$ into $\mathcal{G}_{1,2}$ it follows that $A(\cdot)$ and $\hat{A}(\cdot)$ are both continuous functions into $L^2(M)$ and therefore satisfy the continuity requirement (2.1).

Finally, $A(t)^{g_0} \in C^\infty((0, T) \times M)$ because it is equal to $\hat{A}(t)$ by virtue of (7.9). ■

Remark 7.5 (More on infinite action) In Theorem 7.2 we showed that even if the solution $C(\cdot)$ does not have finite action a weaker version of the ZDS procedure holds. Failure to have finite action can only happen when $a = 1/2$. If $A_0 \in H_{1/2}$ and does not have finite $(1/2)$ -action the conversion function $g(\cdot)$ was only shown to be continuous on $[0, T]$ into the rather large gauge group $\mathcal{G}_{1,2}$ rather than into the natural gauge group $\mathcal{G}_{3/2}$. But the second

order initial behavior bounds for infinite action stated in Theorem 4.16 can be used to show that $g(\cdot)$ is actually continuous on $[0, T]$ into the smaller the gauge group $\mathcal{G}_{1,q}$ for any $q \in [2, 3)$. This would imply that $A(t)$ converges to A_0 in $L^q(M)$ and that $A(t)^{g_0}$ converges to $A_0^{g_0}$ in $L^q(M)$ as $t \downarrow 0$. We will omit here the details of this marginal improvement because the critical value $q = 3$ is still not achieved in the infinite action case.

Proof of Theorems 2.10 and 2.11, Existence. Suppose that $A_0 \in H_a$. If $a > 1/2$ then Theorem 2.20 ensures that there exists a strong solution $C(\cdot)$ to the augmented Yang-Mills heat equation (2.22) on some interval $[0, T]$ with initial value A_0 and satisfying all the hypotheses of Theorem 7.1, which in turn assures the existence of a solution $A(t)$ to (2.5) and a gauge function g_0 satisfying all the conditions required in Theorem 2.10 over the interval $[0, T]$. Since $A(t)^{g_0}$ is a strong solution, it lies in $H_1(M)$ for $t > 0$. Therefore, by [2], it can be extended uniquely to a strong solution over $[0, \infty)$. One can now gauge transform back via g_0^{-1} to find an almost strong solution over all of $[0, \infty)$ which agrees with $A(t)$ for $0 \leq t \leq T$. In this way we have extended the original almost strong solution over $[0, T]$ to an almost strong solution over $[0, \infty)$. This proves items 1) to 5) of Theorem 2.10.

If $a = 1/2$ then Theorem 2.20 ensures that there exists a strong solution $C(\cdot)$ to the augmented Yang-Mills heat equation (2.22) on some interval $[0, T]$ with initial value A_0 and satisfying all the hypotheses of Theorem 7.2, which in turn ensures that there exists a solution $A(\cdot)$ of (2.5) and a gauge function g_0 satisfying the requirements 1) and 2) of Theorem 2.11 after extending the solution to all of $[0, \infty)$ by the method described above. If, moreover, $\|A_0\|_{H_{1/2}}$ is sufficiently small then Theorem 2.20 shows that the solution $C(\cdot)$ will have finite strong $(1/2)$ -action. Theorem 7.1 now ensures that conditions 3) and 4) of Theorem 2.11 also hold.

This concludes the proof of the existence portions of these two theorems. The uniqueness will be proven in Section 7.3. ■

7.2 Initial behavior of A

Notation 7.6 Let $1/2 \leq a < 1$. For a strong solution, $A(\cdot)$, to the Yang-Mills heat equation over $(0, \infty)$ let

$$\rho_a(t) = (1 - a) \int_0^t s^{-a} \|B(s)\|_2^2 ds. \quad (7.18)$$

In accordance with Definition 2.8, a strong solution $A(\cdot)$ has finite a -action in case $\rho_a(t) < \infty$ for some (hence all) $t > 0$.

$\rho_a(t)$ is a gauge invariant function of the initial data A_0 . All of the estimates in this section will be fully gauge invariant. They will depend only on finiteness of $\rho_a(t)$. Finite a -action, as defined by (7.18), is a slightly weaker notion than finite strong a -action, which we have used for $C(\cdot)$, and which is not gauge invariant.

We are going to derive initial behavior estimates of orders one, two and three for a solution $A(\cdot)$ and then apply our Neumann domination techniques from Section 4.7 to derive initial behavior bounds of $\|B(t)\|_\infty$ needed to prove uniqueness of solutions.

Proposition 7.7 (*Order 1*) *If $A(\cdot)$ is a strong solution with finite a -action then*

$$t^{1-a}\|B(t)\|_2^2 + 2 \int_0^t s^{1-a}\|A'(s)\|_2^2 ds = \rho_a(t). \quad (7.19)$$

Proof. For $s > 0$ the identity

$$(d/ds)\|B(s)\|_2^2 = -2\|A'(s)\|_2^2 \quad (7.20)$$

holds, as shown in [2, Equ. (5.7)]. (It is also special case of (4.10) with $\phi = 0$.) In Lemma 4.8 take $f(s) = \|B(s)\|_2^2$, $g(s) = 2\|A'(s)\|_2^2$ and $h(s) = 0$. Then equality holds in (4.28). Choose $b = a$ in Lemma 4.8. Then (4.30) (with equality) asserts that (7.19) holds. ■

Notation 7.8 Recall from (4.2), $\lambda(B(s)) = 1 + \gamma\|B(s)\|_2^4$. We take from [2, Equ. (6.1)] the notation

$$\psi_s^t = 2 \int_s^t \lambda(B(s)) ds \quad \text{and} \quad \psi(t) = \psi_0^t. \quad (7.21)$$

Corollary 7.9 *For $1/2 \leq a < 1$ and $0 < t < \infty$ there holds*

$$t^{2-2a}\|B(t)\|_2^4 \leq \rho_a(t)^2 \quad \text{and} \quad t\|B(t)\|_2^4 \leq t^{2a-1}\rho_a(t)^2, \quad (7.22)$$

$$\int_0^t \|B(s)\|_2^4 ds \leq (1-a)^{-1}t^{2a-1}\rho_a(t)^2, \quad (7.23)$$

$$\sup_{0 < s \leq t} s\lambda(B(s)) < \infty, \quad \text{and} \quad (7.24)$$

$$\psi(t) < \infty. \quad (7.25)$$

Proof. (7.19) shows that

$$\|B(s)\|_2^2 \leq s^{a-1} \rho_a(s). \quad (7.26)$$

Square this to find (7.22). Use it once more to find

$$\begin{aligned} \int_0^t \|B(s)\|_2^4 ds &\leq \int_0^t (s^{a-1} \rho_a(s)) \|B(s)\|_2^2 ds \\ &\leq t^{2a-1} \rho_a(t) \int_0^t s^{-a} \|B(s)\|_2^2 ds, \end{aligned}$$

which, upon using the definition (7.18), gives (7.23). Since $\lambda(B(s)) = 1 + \gamma \|B(s)\|_2^4$, (7.24) and (7.25) follow immediately from (7.22) and (7.23) respectively. ■

Corollary 7.10 *If $A(\cdot)$ is a strong solution with finite a -action then*

$$\begin{aligned} t^{1-a} \|B(t)\|_2^2 + 2\kappa^{-2} \int_0^t s^{1-a} \|B(s)\|_6^2 ds &\leq \rho_a(t) \left(1 + 2 \int_0^t \lambda(B(s)) ds \right) \\ &< \infty. \end{aligned} \quad (7.27)$$

Proof. Since $d_A B = 0$ and $d_A^* B = -A'$ the Gaffney-Friedrichs-Sobolev inequality (4.1) gives

$$\kappa^{-2} \|B(s)\|_6^2 \leq \|A'(s)\|_2^2 + \lambda(B(s)) \|B(s)\|_2^2. \quad (7.28)$$

Therefore

$$2\kappa^{-2} s^{1-a} \|B(s)\|_6^2 \leq 2s^{1-a} \|A'(s)\|_2^2 + 2\lambda(B(s)) (s^{1-a} \|B(s)\|_2^2).$$

But $s^{1-a} \|B(s)\|_2^2 \leq \rho_a(s) \leq \rho_a(t)$ by (7.26). Therefore

$$\begin{aligned} t^{1-a} \|B(t)\|_2^2 + 2\kappa^{-2} \int_0^t s^{1-a} \|B(s)\|_6^2 ds \\ \leq t^{1-a} \|B(t)\|_2^2 + 2 \int_0^t s^{1-a} \|A'(s)\|_2^2 ds + 2\rho_a(t) \int_0^t \lambda(B(s)) ds \\ = \rho_a(t) + 2\rho_a(t) \int_0^t \lambda(B(s)) ds, \end{aligned}$$

which is finite by (7.25). ■

Proposition 7.11 (*Order 2*) *If $A(\cdot)$ is a strong solution with finite a -action then*

$$t^{2-a}\|A'(t)\|_2^2 + \int_0^t s^{2-a} e^{\psi_s^t} \|B'(s)\|_2^2 ds \leq e^{\psi(t)} \rho_a(t). \quad (7.29)$$

Proof. The inequality

$$(d/ds)(e^{-\psi(s)}\|A'(s)\|_2^2) + e^{-\psi(s)}\|B'(s)\|_2^2 \leq 0 \quad (7.30)$$

was proved in [2, Equ. (6.13)]. In Lemma 4.8 take $f(s) = e^{-\psi(s)}\|A'(s)\|_2^2$, $g(s) = e^{-\psi(s)}\|B'(s)\|_2^2$ and $h(s) = 0$. Choose $b = a - 1$. Then (4.28) holds and (4.30) shows that

$$t^{2-a}(e^{-\psi(t)}\|A'(t)\|_2^2) + \int_0^t s^{2-a} e^{-\psi(s)}\|B'(s)\|_2^2 ds \leq (2-a) \int_0^t s^{1-a} e^{-\psi(s)}\|A'(s)\|_2^2 ds.$$

But (7.19) shows that $\int_0^t s^{1-a} e^{-\psi(s)}\|A'(s)\|_2^2 ds \leq (1/2)\rho_a(t)$. Insert this bound into the last displayed inequality and multiply by $e^{\psi(t)}$ to find (7.29). ■

The bounds in the preceding inequalities depend on t and on $\rho_a(t)$. It will be convenient to emphasize this kind of dependence in the following, slightly more complicated inequalities in terms of a standard kind of bounding function. We will call a continuous function from $[0, \infty)^2$ to $[0, \infty)$ a *standard dominating function* if it is zero at $(0, 0)$ and non-decreasing in both arguments. In the following inequalities quantities arising from previous estimates are bounded by standard dominating functions and consequently the new bounds are easily seen to be bounded by new standard dominating functions.

Corollary 7.12 (L^6 estimates.) *If $A(\cdot)$ is a strong solution with finite a -action then*

$$\begin{aligned} t^{2-a}\|B(t)\|_6^2 + \int_0^t s^{2-a} e^{\psi_s^t} \|A'(s)\|_6^2 ds \\ \leq e^{\psi(t)} \rho_a(t) \left(1 + t\lambda(B(t))e^{-\psi(t)} + \int_0^t \lambda(B(s)) ds \right) \\ \leq C_1(t, \rho_a(t)) \end{aligned} \quad (7.31)$$

for some standard dominating function C_1 .

Proof. Since $d_A^* A' = 0$ and $d_A A' = B'$, the Gaffney-Friedrichs-Sobolev inequality (4.1) gives

$$\kappa^{-2} \|A'(s)\|_6^2 \leq \|B'(s)\|_2^2 + \lambda(B(s)) \|A'(s)\|_2^2. \quad (7.32)$$

Therefore, in view of (7.28) and (7.32), we have

$$\begin{aligned} & \kappa^{-2} \left\{ t^{2-a} \|B(t)\|_6^2 + \int_0^t s^{2-a} e^{\psi_s^t} \|A'(s)\|_6^2 ds \right\} \\ & \leq t^{2-a} (\|A'(t)\|_2^2 + \lambda(B(t)) \|B(t)\|_2^2) \\ & \quad + \int_0^t s^{2-a} e^{\psi_s^t} \left(\|B'(s)\|_2^2 + \lambda(B(s)) \|A'(s)\|_2^2 \right) ds \quad (7.33) \\ & \leq e^{\psi(t)} \rho_a(t) + t \lambda(B(t)) (t^{1-a} \|B(t)\|_2^2) + \int_0^t e^{\psi_s^t} \lambda(B(s)) e^{\psi(s)} \rho_a(s) ds. \\ & \leq e^{\psi(t)} \rho_a(t) + t \lambda(B(t)) \rho_a(t) + e^{\psi(t)} \rho_a(t) \int_0^t \lambda(B(s)) ds. \end{aligned}$$

We have applied (7.29) twice to terms in line (7.33), once for dominating the sum of the first and third terms and once for dominating the factor $s^{2-a} \|A'(s)\|_2^2$ in the integral. In the transition to the last line we have used $e^{\psi_s^t} e^{\psi(s)} = e^{\psi(t)}$ along with (7.26). The last line is finite in virtue of (7.24) and (7.25). ■

Corollary 7.13 (*Energy bounds*). *Let $1/2 \leq a < 1$. Then*

$$\int_0^T s^{2-a} \|B(s) \# B(s)\|_2^2 ds < \infty.$$

Proof. Just as in the proof of (4.116) we have the bound

$$\begin{aligned} & s^{2-a} \|B(s) \# B(s)\|_2^2 \\ & \leq c^2 s^{a-(1/2)} \left(s^{(1-a)/2} \|B(s)\|_2 \right) \left(s^{(2-a)/2} \|B(s)\|_6 \right) \left(s^{1-a} \|B(s)\|_6^2 \right). \end{aligned}$$

where c is the commutator bound in \mathfrak{k} . The first two factors in parentheses are bounded by (7.27) and (7.31), respectively. The third factor is integrable by (7.27). ■

Proposition 7.14 (*Neumann Domination*) *Let $1/2 \leq a < 1$. Let $A(\cdot)$ be a strong solution to the Yang-Mills heat equation with finite a -action. Then, for $0 < T < \infty$, there holds*

$$\int_0^T t^{(3/2)-a} \|B(t)\|_\infty^2 ds < \infty. \quad (7.34)$$

In particular,

$$\int_0^T t \|B(t)\|_\infty^2 ds < \infty \quad \text{if } a = 1/2 \quad \text{and} \quad (7.35)$$

$$\int_0^T \|B(t)\|_\infty dt < \infty \quad \text{if } 1/2 < a < 1. \quad (7.36)$$

Further,

$$\|B(t)\|_\infty = o(t^{\frac{a-(1/2)}{2}-1}) \quad \text{as } t \downarrow 0 \quad \text{if } 1/2 \leq a < 1. \quad (7.37)$$

In particular,

$$\|B(t)\|_\infty = o(t^{-1}) \quad \text{as } t \downarrow 0 \quad \text{if } a = 1/2. \quad (7.38)$$

Proof. The Yang-Mills heat equation is a little simpler than the augmented version. The equation (4.8) for B_C can be replaced by

$$B'(s) = \sum_{j=1}^3 (\nabla_j^A)^2 B + B \# B. \quad (7.39)$$

The derivation that led to (4.106) now yields instead

$$|B(t, x)| \leq \frac{1}{t} \int_0^t e^{(t-s)\Delta_N} |B(s, \cdot)| ds(x) + \frac{1}{t} \int_0^t e^{(t-s)\Delta_N} s |B(s) \# B(s)| ds(x).$$

Using $\|e^{(t-s)\Delta_N}\|_{2 \rightarrow \infty} \leq c_1(t-s)^{-3/4}$ it follows that

$$\|B(t)\|_\infty \leq \frac{1}{t} \int_0^t c_1(t-s)^{-3/4} \left(\|B(s)\|_2 + s \|B(s) \# B(s)\|_2 \right) ds. \quad (7.40)$$

Lemma 4.24, with $c = 3/4$, $b = (3/2) - a$ and

$$\beta(s) = c_1 \left(\|B(s)\|_2 + s \|B(s) \# B(s)\|_2 \right),$$

shows that

$$\int_0^T t^{(3/2)-a} \|B(t)\|_\infty^2 dt \leq 2c_1^2 \gamma \int_0^T \left(s^{-a} \|B(s)\|_2^2 + s^{2-a} \|B(s) \# B(s)\|_2^2 \right) ds.$$

It suffices to show therefore that the right hand side is finite. But the integral of the first term is finite by the assumption of finite a-action. The integral of the second term is finite by Corollary 7.13. This proves (7.34). Put $a = 1/2$ in (7.34) to find (7.35). If $a > 1/2$ then

$$\int_0^T \|B(t)\|_\infty dt \leq \left(\int_0^T t^{a-(3/2)} dt \right)^{1/2} \left(\int_0^T t^{(3/2)-a} \|B(t)\|_\infty^2 dt \right)^{1/2} < \infty,$$

which proves (7.36).

To prove (7.37) return to the inequality (4.111) and observe that by (7.19) and (7.31) one has $\|B(s)\|_2 = o(s^{(a-1)/2})$ and $\|B(s)\|_6 = o(s^{(a-2)/2})$, respectively. Since

$$\begin{aligned} \|B(s) \# B(s)\|_2 &\leq c \|B(s)\|_4^2 \leq c \|B(s)\|_2^{1/2} \|B(s)\|_6^{3/2} \\ &= o(s^{(a-1)/4}) o(s^{3(a-2)/4}) = o(s^{a-(7/4)}) \end{aligned}$$

we find

$$s \|B(s) \# B(s)\|_2 = o(s^{a-(3/4)}).$$

Hence

$$\begin{aligned} t \|B(t)\|_\infty &\leq c_1 \int_0^t (t-s)^{-3/4} \left(o(s^{(a-1)/2}) + o(s^{a-(3/4)}) \right) ds \\ &= o(t^{(a-(1/2))/2}) + o(t^{a-(1/2)}) \end{aligned}$$

by (3.30). This proves (7.37). Put $a = 1/2$ in (7.37) to find (7.38). ■

Proposition 7.15 (*Order 3*) For $1/2 \leq a < 1$ and $0 < t < \infty$ there holds

$$t^{3-a} \|B'(t)\|_2^2 + \int_0^t s^{3-a} e^{\psi_s^t} \left(\|A''(s)\|_2^2 + (1/2) \|d_{A(s)}^* B'(s)\|_2^2 \right) ds \leq C_2(t, \rho_a(t)) \quad (7.41)$$

for some standard dominating function C_2 .

The proof depends on the following lemmas.

Lemma 7.16 (*Integral Identity*)

$$\begin{aligned} (d/ds)\|B'(s)\|_2^2 + \|A''(s)\|_2^2 + \|d_{A(s)}^*B'(s)\|_2^2 \\ = \|[A'(s) \lrcorner B(s)]\|_2^2 + 2([A'(s) \wedge A'(s)], B'(s)). \end{aligned} \quad (7.42)$$

Proof. The first two of the identities

$$A'' = -d_A^*B' - [A' \lrcorner B] \quad (7.43)$$

$$B'' = d_A A'' + [A' \wedge A'] \quad (7.44)$$

$$d_A B' = [B \wedge A'] \quad (7.45)$$

follow by differentiating with respect to s , first the Yang-Mills heat equation itself and then the identity $B' = d_A A'$. The third follows from Bianchi's identity: $d_A B' = (d_A)^2 A' = [B \wedge A']$. From (7.44) we find that

$$\begin{aligned} (d/ds)\|B'(s)\|_{L^2}^2 &= 2(B'', B') \\ &= 2(d_A A'' + [A' \wedge A'], B') \\ &= 2(A'', d_A^* B') + 2([A' \wedge A'], B'). \end{aligned}$$

We may evaluate the first term on the right in two different ways: Replace $d_A^* B'$ using (7.43) or replace A'' using (7.43). We find $(A'', -A'' - [A' \lrcorner B]) = (A'', d_A^* B') = (-d_A^* B' - [A' \lrcorner B], d_A^* B')$. Adding these two representations we find

$$\begin{aligned} 2(A'', d_A^* B') &= -\|A''\|_2^2 - \|d_A^* B'\|_2^2 - (A'' + d_A^* B', [A' \lrcorner B]) \\ &= -\|A''\|_2^2 - \|d_A^* B'\|_2^2 + \|A' \lrcorner B\|_2^2. \end{aligned}$$

This proves (7.42). ■

Lemma 7.17 (*Differential inequality, order three.*)

$$\begin{aligned} (d/ds)\|B'(s)\|_2^2 + \|A''(s)\|_2^2 + (1/2)\|d_{A(s)}^*B'(s)\|_2^2 \\ \leq (3c^2/2)\|B(s)\|_6^2\|A'(s)\|_3^2 + 2(c\kappa)^2\|A'(s)\|_2^2\|A'(s)\|_3^2 \\ + (1/2)\lambda(B(s))\|B'(s)\|_2^2. \end{aligned} \quad (7.46)$$

Proof. We need only find appropriate bounds for the terms on the right side of (7.42). For the first term we have the simple Hölder bound $\| [A' \lrcorner B] \|_2^2 \leq c^2 \|B\|_6^2 \|A'\|_3^2$.

Concerning the second term in (7.42) we may apply Hölder and then the Gaffney-Friedrichs-Sobolev inequality (4.1) to find

$$\begin{aligned}
2|([A' \wedge A'], B')| &\leq 2c \int_M |A'| |A'| |B'| dx \\
&\leq 2c \|A'\|_2 \|A'\|_3 \|B'\|_6 \\
&\leq (1/2) \left(2c\kappa \|A'\|_2 \|A'\|_3 \right)^2 + (1/2) \kappa^{-2} \|B'\|_6^2 \\
&\leq 2(c\kappa)^2 \|A'\|_2^2 \|A'\|_3^2 + (1/2) \left(\|d_A^* B'\|_2^2 + \|d_A B'\|_2^2 + \lambda(B) \|B'\|_2^2 \right) \\
&= 2(c\kappa)^2 \|A'\|_2^2 \|A'\|_3^2 + (1/2) \|d_A^* B'\|_2^2 \\
&\quad + (1/2) \| [B \wedge A'] \|_2^2 + (1/2) \lambda(B) \|B'\|_2^2,
\end{aligned}$$

wherein we have used (7.45). We can cancel $(1/2) \|d_A^* B'\|_2^2$ with a half of the corresponding term on the left side on (7.42). Using $\| [A' \lrcorner B] \|_2^2 + (1/2) \| [B \wedge A'] \|_2^2 \leq c^2 (3/2) \|B\|_6^2 \|A'\|_3^2$ we arrive at (7.46). ■

Lemma 7.18 *There are constants c_7, c_8 depending only on Sobolev constants and the commutator bound c such that*

$$\begin{aligned}
(d/ds) \left(e^{-\psi(s)} \|B'(s)\|_2^2 \right) + e^{-\psi(s)} \left(\|A''(s)\|_2^2 + (1/2) \|d_{A(s)}^* B'(s)\|_2^2 \right) \\
\leq e^{-\psi(s)} \left\{ c_7 \|B(s)\|_6^2 \|A'(s)\|_3^2 + c_8 \|A'(s)\|_2^2 \|A'(s)\|_3^2 \right\}. \quad (7.47)
\end{aligned}$$

Proof. The first term is $e^{-\psi(s)} \left((d/ds) \|B'(s)\|_2^2 - \lambda(B(s)) \|B'(s)\|_2^2 \right)$. Therefore multiplication of (7.46) by $e^{-\psi(s)}$ yields (7.47) if one chooses $c_7 = 3c^2/2$ and $c_8 = 2c^2\kappa^2$. ■

Proof of Proposition 7.15. We will apply Lemma 4.8 with $f(s) = e^{-\psi(s)} \|B'(s)\|_2^2$, $g(s) = e^{-\psi(s)} \left(\|A''(s)\|_2^2 + (1/2) \|d_{A(s)}^* B'(s)\|_2^2 \right)$ and $h(s)$ equal to the entire right hand side of (7.47). Then (4.28) holds in virtue of (7.47).

Choose $1 - b = 3 - a$, i.e. $b = a - 2$. Then (4.30) shows that

$$\begin{aligned} & t^{3-a} e^{-\psi(t)} \|B'(t)\|_2^2 + \int_0^t s^{3-a} e^{-\psi(s)} \left(\|A''(s)\|_2^2 + (1/2) \|d_{A(s)}^* B'(s)\|_2^2 \right) ds \\ & \leq (3-a) \int_0^t s^{2-a} e^{-\psi(s)} \|B'(s)\|_2^2 ds \\ & + \int_0^t s^{3-a} e^{-\psi(s)} \left\{ c_7 \|B(s)\|_6^2 \|A'(s)\|_3^2 + c_8 \|A'(s)\|_2^2 \|A'(s)\|_3^2 \right\} ds. \quad (7.48) \end{aligned}$$

The first integral on the right is finite by (7.29) and justifies use of Lemma 4.8. Upon multiplying (7.48) by $e^{\psi(t)}$ we find an inequality whose left side is the left side of (7.41). It remains to show that the last integral in line (7.48) is finite. From our bounds (7.29) and (7.31) of order two we have

$$\eta \equiv \sup_{0 < s \leq t} s^{2-a} \left(c_7 \|B(s)\|_6^2 + c_8 \|A'(s)\|_2^2 \right) < \infty.$$

Therefore the integral in line (7.48) is at most

$$\begin{aligned} \eta \int_0^t s e^{-\psi(s)} \|A'(s)\|_3^2 ds & \leq \eta \int_0^t s^{1/4} \|A'(s)\|_2 s^{3/4} \|A'(s)\|_6 ds \\ & \leq \eta \left(\int_0^t s^{1/2} \|A'(s)\|_2^2 ds \right)^{1/2} \left(\int_0^t s^{3/2} \|A'(s)\|_6^2 ds \right)^{1/2} \\ & < \infty, \end{aligned}$$

wherein we have used (7.19) and (7.31), with $a = 1/2$, which is allowed because $\rho_{1/2}(t) < \infty$ if $\rho_a(t) < \infty$ for some $a \geq 1/2$. ■

Corollary 7.19 *For $1/2 \leq a < 1$ and $0 < t < \infty$ there holds*

$$t^{3-a} \|A'(t)\|_6^2 \leq C_3(t, \rho_a(t)). \quad (7.49)$$

for some standard dominating function C_3 .

Proof. Since $d_A^* A' = -d_A^*(d_A^* B) = 0$, the Gaffney-Friedrichs-Sobolev inequality (4.1) gives

$$\begin{aligned} \kappa^{-2} \|A'(t)\|_6^2 & \leq \|d_A A'\|_2^2 + \lambda(B(t)) \|A'(t)\|_2^2 \\ & \leq \|B'(t)\|_2^2 + (1 + \gamma \|B(t)\|_2^4) \|A'(t)\|_2^2. \end{aligned}$$

We see from (7.41) that $t^{3-a} \|B'(t)\|_2^2$ is bounded. Moreover (7.29) shows that $t^{2-a} \|A'(t)\|_2^2$ is also bounded. Since, by (7.24), $t \|B(t)\|_2^4$ is also bounded, the inequality (7.49) follows. ■

7.3 Uniqueness of A

Theorem 7.20 (*Uniqueness for $a = 1/2$.*) Suppose that $A_1(\cdot)$ and $A_2(\cdot)$ are two strong solutions with finite action and the same initial value. Assume that, if $M \neq \mathbb{R}^3$, then for all $t > 0$, both satisfy the boundary conditions (7.50) in case (N) or (7.51) in case (D).

$$B_j(t)_{norm} = 0 \quad \text{in case (N)} \quad (7.50)$$

$$A_j(t)_{tan} = 0 \quad \text{in case (D)} \quad (7.51)$$

Then $A_1(t) = A_2(t)$ for all $t \geq 0$.

The proof will require the following lemma.

Lemma 7.21 If $A_j(\cdot)$, $j = 1, 2$, are two strong solutions of finite action with the same initial value then

$$\|A_1(t) - A_2(t)\|_2^2 = o(t^{1/2}) \text{ as } t \downarrow 0. \quad (7.52)$$

Proof. Since

$$\begin{aligned} \|A_1(t) - A_2(t)\|_2 &\leq \|A_1(t) - A_0 + A_0 - A_2(t)\|_2 \\ &\leq \|A_1(t) - A_0\|_2 + \|A_0 - A_2(t)\|_2, \end{aligned}$$

it suffices to show that each term is $o(t^{1/4})$. For any solution $A(\cdot)$ of finite action one has

$$\begin{aligned} \|A(t) - A_0\|_2 &\leq \int_0^t \|A'(s)\|_2 ds \\ &= \int_0^t s^{-1/4} \left(s^{1/4} \|A'(s)\|_2 \right) ds \\ &\leq \left(\int_0^t s^{-1/2} ds \right)^{1/2} \left(\int_0^t s^{1/2} \|A'(s)\|_2^2 ds \right)^{1/2} \\ &= t^{1/4} \sqrt{2} \left(\int_0^t s^{1/2} \|A'(s)\|_2^2 ds \right)^{1/2}. \end{aligned}$$

The integral is finite by the energy estimate (7.19) (with $a = 1/2$) and therefore the integral is $o(1)$ as $t \downarrow 0$. ■

Proof of Theorem 7.20. The identity [2, Equ. (8.63)] shows that

$$\frac{d}{dt} \|A_1(t) - A_2(t)\|_2^2 \leq c(\|B_1(t)\|_\infty + \|B_2(t)\|_\infty) \|A_1(t) - A_2(t)\|_2^2. \quad (7.53)$$

This was derived in [2] in case M is compact. The proof in case $M = \mathbb{R}^3$ is easier since one need not be concerned with boundary conditions. We omit the minor changes.

Let $f(t) = \|A_1(t) - A_2(t)\|_2^2$ and $u(t) = c(\|B_1(t)\|_\infty + \|B_2(t)\|_\infty)$. Then

$$f'(t) \leq u(t)f(t), \quad t > 0 \quad (7.54)$$

for $t > 0$ by (7.53). f is continuous on $[0, T]$ because each $A_j(t)$ converges to A_0 in $L^2(M)$ as $t \downarrow 0$. Since $f(0) = 0$ it follows that

$$\begin{aligned} f(t) &= \int_0^t f'(s) ds \\ &\leq \int_0^t u(s)f(s) ds \\ &\leq \left(\int_0^t su(s)^2 ds \right)^{1/2} \left(\int_0^t s^{-1} f(s)^2 ds \right)^{1/2}. \end{aligned} \quad (7.55)$$

Let $g(t) = f(t)/\sqrt{t}$ for $t > 0$. By Lemma 7.21 we see that g is bounded on $(0, T]$ and in fact goes to zero as $t \downarrow 0$. For convenience we may extend g continuously to $[0, T]$ by defining $g(0) = 0$. Let

$$w(t) = \left(\int_0^t su(s)^2 ds \right)^{1/2}. \quad (7.56)$$

Then $w(t) < \infty$ for $0 \leq t \leq T$ by (7.35). Dividing (7.55) by \sqrt{t} we find

$$g(t) \leq w(t) \left(\frac{1}{t} \int_0^t g(s)^2 ds \right)^{1/2}. \quad (7.57)$$

There is a constant C such that $g(t) \leq C$ for $0 \leq t \leq T$. Insert this bound in the integral in (7.57) to find that $g(t) \leq w(t)C$. We can now proceed by induction using the fact that w is non-decreasing: Assuming that $g(s) \leq w(s)^n C$ for $0 \leq s \leq T$, (7.57) then implies that

$$\begin{aligned} g(t) &\leq w(t) \left(\frac{1}{t} \int_0^t w(s)^{2n} C^2 ds \right)^{1/2} \\ &\leq w(t) \{w(t)^n C\}. \end{aligned}$$

Consequently $g(t) \leq w(t)^{n+1}C$. Thus if $t_0 > 0$ is such that $w(t) \leq 1/2$ for $0 \leq t \leq t_0$ then $g(t) = 0$ on $[0, t_0]$. Hence $A_1(t) = A_2(t)$ on this interval. Since $A_j(t) \in H_1(M)$ for $j = 1, 2$ and all $t > 0$ we can now use the uniqueness theorem in [2] for H_1 initial data to conclude that $A_1(t) = A_2(t)$ for all $t > 0$. ■

Remark 7.22 (Uniqueness for $a > 1/2$) If a solution to the Yang-Mills heat equation has finite a -action for some $a \geq 1/2$ then it has finite $(1/2)$ -action, as is clear from the definition (7.18). Our uniqueness proof applies therefore to all $a \in [1/2, 1)$. However if $a > 1/2$ then the inequality (7.35), on which our proof rests, can be replaced by (7.36). Thus for $a > 1/2$ we have $\int_0^t \|B_j(s)\|_\infty ds < \infty$, $j = 1, 2$ by the apriori bound (7.36). The function u that appears in (7.54) is therefore integrable over $[0, T]$. Consequently the standard Gronwall argument for uniqueness is applicable: the non-negative function $h(t) \equiv e^{-\int_0^t u(s)ds} f(t)$ has a non-positive derivative on $(0, T]$ and is zero at $t = 0$, hence is identically zero on $[0, T]$. This is the basis for the uniqueness proof used in [2] for the case of finite energy ($a = 1$). Here we see another instance of breakdown of standard techniques at criticality.

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